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ABSTRACT

This book is designed to introduce the reader to some fundamental ideas about probability. It is assumed that the reader has completed chapters 1 to 7 of SMSG Introduction to Probability, Part 1. While the book is designed for junior high school students, some elementary algebra is required for a number of the exercises. The difficult exercises are marked to assist students and teachers. Chapters included are: (8) Bayes' Formula; (9) Bernoulli Trials; (10) Mathematical Expectation; (11) Bertrand's Ballot Problem; and (12) Markov Chains. Answers to exercises and additional materials are included in the appendices. (RH)

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SCHOOL
MATHEMATICS
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INTRODUCTION TO PROBABILITY

Part 2 – Special Topics

Student Text

(Revised Edition)

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2

INTRODUCTION TO PROBABILITY

Part 2 — Special Topics

Student Text

(Revised Edition)

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PREFACE

For Students and Teachers

Mathematical Background

It is assumed, throughout this volume, that the reader has completed Chapters 1 to 6 of SMSG INTRODUCTION TO PROBABILITY - Part I. In addition, certain topics depend heavily on the ideas about conditional probability developed in Chapter 7 of Part I. Others require some familiarity with certain ideas and skills of elementary algebra.

The more difficult sections of the text are marked T, as are some of the more demanding exercises.

Outline of Content

Chapter 8 develops techniques for applying the results of Chapter 7 and, hence, depends on the latter. Section 8-1 is not very difficult, and it supplies the student with a useful method of attacking problems which seem complicated. The exercises of Section 8-2 illustrate the wide range of such problems. Bayes' formula, which is discussed in Sections 8-3 and 8-4, is essentially an algebraic restatement and generalization of earlier results on conditional probability. It should be omitted by students who have not had some experience in algebra.

The material on Bernoulli trials in Chapter 9 may be studied immediately following the completion of Chapter 6. It should be noted, however, that experience with Section 8-1 will make the understanding of Chapter 9 easier. The content of Sections 9-6 and 9-8 will be meaningful only for students with some background in algebra. Chapter 9 does not include a complete treatment of permutations and combinations, and it requires no previous acquaintance with this topic. However, students who have encountered permutations and combinations elsewhere will be able to apply their knowledge here. For many problems in probability more than one method can be used, and it is both interesting and instructive to see how different approaches lead to the same result.

The first five sections of Chapter 10 are not particularly difficult. They could be studied immediately after Chapter 6. Sections 10-6 through 10-8 introduce ideas that are important for the study of statistics. These sections would be of particular interest to students who are using empirical data in a natural science or social studies course. Section 10-9 requires some knowledge of algebra. Section 10-10 is a rather lengthy application of ideas of Chapter 9.

Bertrand's ballot problem (Chapter 11) can be studied independently of the rest of the volume. It is an interesting problem and gives students an opportunity to enjoy the pleasures of discovery. The material in Sections 11-1 through 11-5 is relatively easy and most students should be able to complete it readily. Section 11-6 shows how conditional probability--and Bayes' formula--enters the problem.

Chapter 12 (Markov Chains) applies the ideas of conditional probability and should not be attempted by those who omit Section 8-1. The ideas here are somewhat more difficult but they are extremely useful in many practical situations. Incidentally, the reader of Chapter 12 encounters repeating decimals in a rather interesting setting.

Suggested Plans of Study

1. For those who omit Chapter 7 (Part I) the following is suggested:

   Chapter 9, Sections 1, 2, 3, 4 and 5.
   Chapter 10, Sections 1, 2, 3, 4 and 5.
   Chapter 11, Sections 1, 2, 3, 4 and 5. (Optional)

2. A minimal program for those who complete Chapter 7 would include the list above, plus Chapter 8, Sections 1 and 2, and Chapter 12, as time permits.

3. Students who have studied (or who are studying) algebra might profit by completing Chapter 9, omitting perhaps either Chapter 11 or 12.

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Chapter 8

BAYES' FORMULA

8-1. Tree Diagrams

The material of this chapter depends heavily on the concept of conditional probability. In particular, we shall make use of the formula from Chapter 7.

(1) $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$

If we know $P(E\cap F)$ and $P(F)$, we use formula (1) to find $P(E|F)$.

$.6\left(= \frac{.3}{.5}\right)$

1. For example, if $P(E\cap F) = .3$, $P(F) = .5$, then $P(E|F) =$ ______.

It may happen, however, that we know $P(E|F)$ and $P(F)$ and wish to find $P(E\cap F)$.

2. If $P(E|F) = \frac{2}{3}$, $P(F) = \frac{1}{2}$, then

$$\frac{2}{3} = \frac{P(E\cap F)}{\frac{1}{2}} .$$

$\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

3. So, $P(E\cap F) = \frac{1}{2} \cdot \quad =$ ______.

$P(F) \cdot P(E|F)$

4. $P(E|F) = \dfrac{P(E\cap F)}{P(F)}$ may be written in the form:

$$P(E\cap F) = P(F) \cdot \underline{\qquad\qquad} .$$

This form is useful in many situations.

Let us start with an experiment.

Experiment:

Use one coin and one die. Toss the coin.

(1) If heads occurs, record "I". Then throw the die. If 1, 2, 3 or 4 occurs, record "R"; for 5 or 6, record "G".

(2) If tails occurs, record "II". Then throw the die. If 1, 3 or 5 occurs, record "R". If 2, 4 or 6 occurs, record "G".

If you do the experiment several times, what fraction of the time do you expect "I" ? "II" ? "R" ? "B" ? Of those trials when "I" is recorded, what fraction of the time do you expect "R" ? "G" ?

Perform 30 trials of the experiment. A table is useful in keeping track of the results.

	Number Recorded		Number Recorded
I		R	
		B	
II		R	
		B	

Our results and a brief discussion of this experiment are on page 305.

$\frac{1}{4}(= \frac{1}{2} \cdot \frac{1}{2})$

5. If $P(R|II) = \frac{1}{2}$, $P(II) = \frac{1}{2}$, then $P(II \cap R) =$ _____ regardless of what events "II" and "R" indicate.

In Items 4 and 5 we have used the formula:

$$(2) \quad P(E \cap F) = P(F) \cdot P(E|F).$$

Formula (2) is a general formula for obtaining $P(E \cap F)$ whenever we know $P(F)$, $P(E|F)$. You may recall that the question of $P(E \cap F)$ was considered briefly in Chapter 5. At that point we were only able to deal with the case of mutually exclusive events, in which case $P(E \cap F) = 0$. In Chapter 6 we introduced the notion of independent events. If E, F are independent, we have $P(E \cap F) = P(E) \cdot P(F)$. You should notice now that this last may be thought of as a special case of (2). Independent events were discussed in more detail in Section 7-6.

$P(E)$

6. If E, F are independent events, then $P(E|F) =$ _____. (Section 7-6)

$P(F) \cdot P(E)$

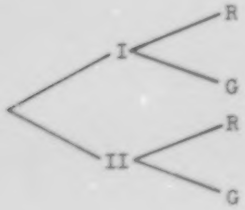
7. Hence, $P(E \cap F) = P(F) \cdot P(E|F)$ becomes $P(E \cap F) =$ _____.

With this short review behind us, let us see how we may use formula (2) in connection with tree diagrams.

You have become familiar with tree diagrams as an aid in counting outcomes. In this section we shall see how tree diagrams are useful in calculating probabilities in a variety of situations.

Consider the following experiment:

> There are two urns*. Urn I contains two red marbles and one green marble. Urn II contains one red and one blue marble. An urn is to be selected at random. (Recall that "at random" means that each is equally likely to be selected.) A marble is then drawn (again, at random) from the selected urn.
>
> An obvious set of outcomes for this experiment is:
>
> $$\{I \cap R\ ,\ I \cap G\ ,\ II \cap R\ ,\ II \cap G\}.$$
>
> By $I \cap R$ we mean that Urn I is selected and that a red marble is drawn. The following tree diagram fits this experiment:



You recognize the similarity between this "urn problem" and the coin and die experiment that you just completed.

The given information enables us to write down immediately many of the probabilities we need.

---

[*] In the literature of probability, containers for marbles, balls, numbered chips, etc., have always been referred to as "urns". A problem, such as the present one, is called an "urn problem". Urn problems are useful as models or examples which simulate a variety of practical applications of probability theory.

$\frac{1}{2}$ 8. $P(I) =$ ______. "I" is the event "Urn I is selected".)

$\frac{1}{2}$ $P(II) =$ ______.

$\frac{2}{3}$ $P(R|I) =$ ______.

$\frac{1}{3}$ $P(G|I) =$ ______.

$\frac{1}{2}$ $P(R|II) =$ ______.

$\frac{1}{2}$ $P(G|II) =$ ______.

Here, then, is a situation where we know $P(I)$, $P(II)$ and certain conditional probabilities. It is natural to use formula (2) to find the probabilities of $I \cap R$, $I \cap G$, etc.

9. $P(I \cap R) = P(I) \cdot P(R|I)$, using formula (2). Hence,

$\frac{1}{3}$ $P(I \cap R) = \frac{1}{2} \cdot \frac{2}{3} =$ ______.

10. Similarly:

$\frac{1}{6}$ $P(I \cap G) =$ ______.

$\frac{1}{4}$ $P(II \cap R) =$ ______. (We obtained this result in Item 5.)

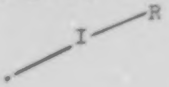
$\frac{1}{4}$ $P(II \cap G) =$ ______.

11. Of course,

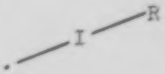
$1 (= \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{4})$ $P(I \cap R) + P(I \cap G) + P(II \cap R) + P(II \cap G) =$ ______.

This last result is not surprising. Our set of outcomes is $\{I \cap R, I \cap G, II \cap R, II \cap G\}$.

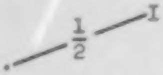
Let us go back to our tree diagram. We will look at each branch individually. If we follow the branch



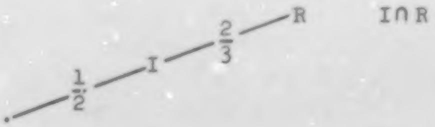
, we interpret this as "Urn I is chosen and a red marble is drawn". We may, then, label the branch as $I \cap R$.



$I \cap R$

On the particular piece, , we shall write "$\frac{1}{2}$". Since $P(I) = \frac{1}{2}$: . That is, we label the piece with "its" probability.

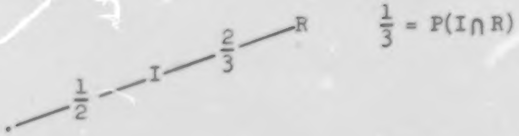
How about the piece, ? What probability label shall we give it? If you think a moment, you will agree that the appropriate probability is "the probability of R, given that urn I is selected". Since $P(R|I) = \frac{2}{3}$, we now have:



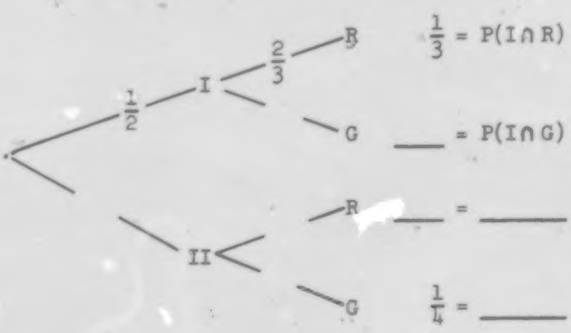
But, $P(I\cap R) = P(I)\cdot P(R|I)$

$$= \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}\,.$$

To find the probability of the branch "I∩R", we multiply the probabilities that occur along that branch.



12. Complete the tree diagram below by supplying the appropriate probabilities:

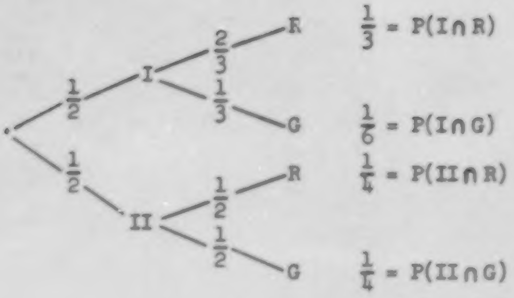


The completed diagram is shown on the following page.

$\frac{1}{6}$ = P(I∩G)

$\frac{1}{4}$ = P(II∩R)

$\frac{1}{4}$ = P(II∩G)



It is important to notice several things about this diagram.

(a) The probabilities on the second-step pieces are conditional probabilities:

$$P(R|I)\ ,\ P(G|I)\ ,\ P(R|II)\ ,\ P(G|II).$$

(b) Each complete branch represents one of the outcomes of our original set of outcomes: $\{I \cap R\ ,\ I \cap G\ ,\ II \cap R\ ,\ II \cap G\}$.

(c) The probabilities of each complete branch are obtained by multiplying the probabilities that occur along the branch.

(d) The sum of the probabilities of the complete branches is 1.

We can use our results to find other probabilities. In our example, the event "red" can only occur in one of two mutually exclusive ways, as $I \cap R$ or as $II \cap R$.

+, ($I \cap R$ and $II \cap R$ are mutually exclusive)	13. $P(R) = P(I \cap R) \underset{(+,\ \cdot)}{\underline{\qquad}} P(II \cap R)$.
$\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$	14. $P(R) = \frac{1}{3} + \underline{\qquad} = \underline{\qquad}$.
$P(I \cap G) +$ $P(II \cap G)$	15. Similarly, $P(G) = \underline{P(\qquad) + P(\qquad)}$.
$\frac{1}{6} + \frac{1}{4} = \frac{5}{12}$	16. $P(G) = \underline{\qquad} + \frac{1}{4} = \underline{\qquad}$.
1 (naturally!)	17. $P(R) + P(G) = \underline{\qquad}$.

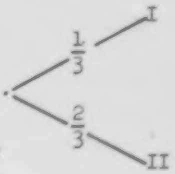
The next paragraph deals with another urn problem, similar to the preceeding example. Try to work this problem by yourself. Sketch a tree diagram, labeling the branches with the appropriate probabilities. Compare your answers with Item 34. A step-by-step solution is given in Items 18 to 33. Before you begin, notice that the given information enables you immediately to write the values of P(I) , P(II). Also obvious are the conditional probabilities that a particular color is drawn, given that a certain urn is selected.

Urn I contains 5 red, 3 white, and 2 blue marbles. Urn II contains 3 red and 7 blue marbles. We throw a die to determine which urn to select. If the die shows "1" or "2", we use Urn I, otherwise Urn II. A marble is drawn at random from the chosen urn. Find P(R) , P(W) , P(B).

$\frac{1}{3}$ 18. P(I) = ______, since, for the die, P(1 or 2) = $\frac{1}{3}$.

$\frac{2}{3}$ 19. P(II) = ______.

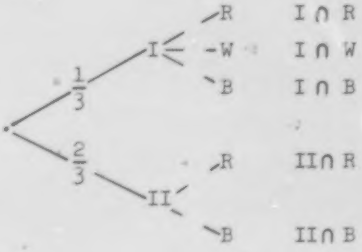
Our tree diagram starts like this:

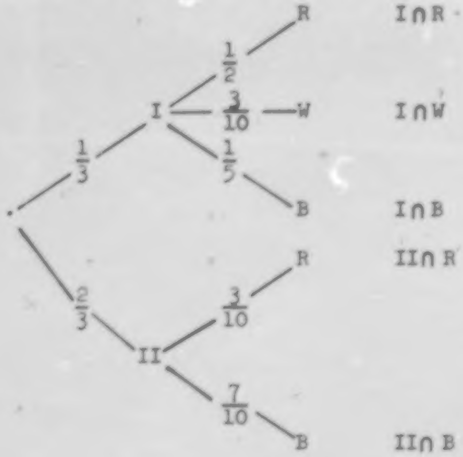


3 20. From the point labeled "I", we need ______ (how many) branches, one each for red, white and blue.

2 21. From the point labeled "II", we need only ______ (how many) branches.

We have, then,



Urn I	22. $P(R \mid I)$ means the probability of drawing a red marble given that ______ is selected.
$\frac{1}{2}$ (or $\frac{5}{10}$) $\frac{3}{10}$ $\frac{1}{5}$ (or $\frac{2}{10}$) $\frac{3}{10}$ $\frac{7}{10}$	23. Since Urn I contains 5 red, 3 white, and 2 blue marbles, all equally likely, we have: $P(R \mid I)$ = ______, $R(W \mid I)$ = ______, $R(B \mid I)$ = ______. In the same way, $P(R \mid II)$ = ______, $P(B \mid II)$ = ______. Our tree diagram now looks like this:  R W B R B I II 1/2 3/10 1/5 1/3 2/3 3/10 7/10 I∩R I∩W I∩B II∩R II∩B We are now ready to compute the probabilities of $I \cap R$, $I \cap W$, etc.
multiply	24. To find $P(I \cap R)$, we ______ (add, multiply) $\frac{1}{3}$ and $\frac{1}{2}$.
$\frac{1}{6}$	25. $P(I \cap R)$ = ______.
$\frac{1}{10}(= \frac{1}{3} \cdot \frac{3}{10})$	26. $P(I \cap W)$ = ______.
$\frac{1}{15}(= \frac{1}{3} \cdot \frac{1}{5})$	27. $P(I \cap B)$ = ______.

$\frac{1}{5}(=\frac{2}{3}\cdot\frac{3}{10})$	28. $P(II \cap R) =$ ______.
$\frac{7}{15}(=\frac{2}{3}\cdot\frac{7}{10})$	29. $P(II \cap B) =$ ______.
	Now, the event "red marble" can occur in one of two mutually exclusive ways. Either the red marble comes from Urn I or from Urn II.
+	30. $P(R) = P(I \cap R) \underset{(\cdot\ ,\ +)}{\underline{\qquad}} P(II \cap R)$.
$\frac{1}{6}+\frac{1}{5}$	31. $P(R) = \frac{1}{6} +$ ______ (Items 25, 28)
$\frac{11}{30}$	$= \frac{\square}{30}$.
	32. By a similar argument,
$\frac{1}{15}$	$P(B) =$ ______ $+ \frac{7}{15}$ (Items 27, 29)
$\frac{8}{15}$	$= \frac{\square}{15}$
$\frac{1}{10}$	33. $P(W) = \frac{\square}{10}$ (Item 26)
	34. Changing our answers to fractions with denominator 30,
$\frac{11}{30}$	$P(R) =$ ______,
$\frac{3}{30}$	$P(W) =$ ______,
$\frac{16}{30}$	$P(B) =$ ______.
	35. We should be careful to observe:
1	$P(R) + P(W) + P(B) = \frac{11}{30} + \frac{3}{30} + \frac{16}{30} =$ ______.

Let us briefly review.

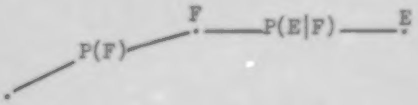
From

$$(1)\quad P(E|F) = \frac{P(E \cap F)}{P(F)}$$

we may conclude:

$$(2) \quad P(E \cap F) = P(F) \cdot P(E|F) .$$

We often use this multiplication formula in connection with tree diagrams. A typical branch might look like this:



$$P(E \cap F) = P(F) \cdot P(E|F)$$

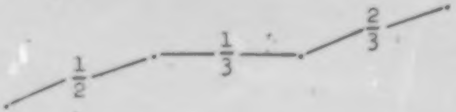
Since each complete branch of a tree diagram represents one of a set of mutually exclusive events, we may add the probabilities at the tips of different branches to obtain a desired probability. Suppose, for example, that three branches lead to E. That is, E occurs in $E \cap F_1$, $E \cap F_2$, $E \cap F_3$. Our formulas would then become:

$$P(E) = P(E \cap F_1) + P(E \cap F_2) + P(E \cap F_3)$$

and

$$P(E) = P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + P(F_3) \cdot P(E|F_3) .$$

It should also be apparent how we would extend the diagram if we had a situation involving more than two steps. For example, the probability of a branch such as



is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{9}$.

The exercises of Section 8-2 are designed to give you a variety of practice in using tree diagrams.

8-2. Exercises. (Answers on page 314.)

1. Urn problem. There are three urns, I, II, III.

   Urn I contains three chips, numbered 1, 2, 3.
   Urn II contains two chips, numbered 1, 2.
   Urn III contains two chips, numbered 2, 4.

   An urn is chosen at random and a chip is drawn at random. What is the probability that the chip drawn is numbered 1 ? 2 ? 3 ? 4 ?

2. Referring to Exercise 1, find P(chip has an even number), P(chip has a number less than 3), and P(chip is even or less than 3).

3. Seeing two gum machines, a boy doesn't know which to use. He flips a coin to decide. It happens that machine A gives 3 pieces with probability $\frac{1}{5}$, and 1 piece with probability $\frac{4}{5}$. Machine B gives 1 or 2 pieces equally often. Find the probabilities that the boy receives 1, 2, 3 pieces.

4. Here is another urn problem, this time using only one urn. The urn contains 6 red, 4 blue marbles. A marble is drawn, its color noted, and then replaced. A second draw is then made and the color recorded. (This situation--first discussed in Chapter 6--is one of "drawing with replacement". The events "red on first draw", "red on second draw" are independent.) The number of times that red is drawn may be either 0, 1 or 2. Find P(0), P(1), and P(2).

5. Using the urn of Exercise 4, we again draw twice. This time, however, we do not replace the first marble. Find P(0), P(1), P(2). Hint: After the first draw there are only 9 marbles left.

6. Under the conditions of Exercise 4 (two draws with replacement), find P(red on second draw).

7. Under the conditions of Exercise 5 (two draws without replacement), find P(red on second draw).

8. Kate and Jane play a simple game. Kate has two disks, each one red on one side and green on the other. Jane has one such disk. At a given signal Jane and Kate each put a disk on the table. If they show the same color, Kate takes both of them; if the colors are different, Jane takes both of them. They play until one player has no more disks or until they have

compared disks three times. Make a tree to show the progress of the game. We know that the probability that Jane wins any particular play is $\frac{1}{2}$. Find the following probabilities:

(a) Jane wins the game;

(b) Kate wins;

(c) neither wins.

9. In the game of Exercise 8, there are 3 disks in all. Find the probability that Kate ends up with 0 disks, 1 disk, 2 disks, 3 disks.

10. In tennis, a player must win 6 games to win the set, but he must lead his opponent by 2 games when he has won the 6 games. Otherwise, the set continues until one or the other player has a two-game advantage. Art and Bill are playing a set. After 8 games the score is 4-4. Bill has just hurt his hand so that from now on the probability that Art will win any game is $\frac{2}{3}$. They agree to play until either someone wins (6-4 or 7-5) or to settle for a tie if the score reaches 6-6.

    Find the probabilities: P(Art wins), P(Bill wins), P(tie).

11. Here is a game which you might play. An urn contains 7 red and 3 green balls. You are to select a ball, note its color and replace it. Your opponent is then to select a ball. R_y is the event that you select red; R_o is the event that your opponent selects red; G_y is the event that you select green, etc. Make a tree diagram of possible outcomes. You win if the ball your opponent selects has the same color as the ball you selected. What is your probability of winning?

12. Our next problem is a little different from the other urn problems. We are going to draw a marble, note its color, replace it, and then add two more of the same color. We have an urn with 6 red and 4 blue marbles. We select a marble at random, note its color, and return it and add 2 more of the same color to the urn. We repeat the procedure again. What is the probability of selecting a red marble on the second drawing? The probability of a red marble on the first drawing is $\frac{3}{5}$. Do you think the probability of a red marble on subsequent draws will be greater or less than $\frac{3}{5}$?

†13. (a) In Exercise 12, suppose we continue the same procedure (replace the marble and add two of that color). Find P(red on third draw).

(b) Can you guess at a generalization? What if the original distribution of marbles had been 5 red, 5 green? r red and g green?

14. A certain cook can prepare two cereals, Lumpies and Soggies, but sometimes she burns them. In fact, when she cooks Lumpies, her probability of burning it is .1. Whenever she burns Lumpies, then she cooks Soggies the next day. However, she really doesn't like Soggies very well, even when it isn't burned. Consequently, after cooking it one day, she always goes back to Lumpies. She begins a new job on Monday morning by cooking Lumpies. What is the probability that on Wednesday she cooks Soggies? Lumpies?

†15. Our cook is even more careless with Soggies. Her probability of burning Soggies whenever she cooks them is .4. What is the probability that Wednesday's cereal is burned?

†16. The careless cook finds that she has grown to like Soggies, so she changes her plan of operation. She begins, on Monday, January 1, another year, by cooking Lumpies. Again, she cooks Lumpies until she burns it, and then changes to Soggies. Now, however, she cooks Soggies until she burns that, and then changes back to Lumpies again. Unfortunately, in all this time her cereal-cooking has not improved. Her probability of burning Lumpies if she cooks it is .1, and her probability of burning Soggies if she cooks it is .4.

(a) What is the probability that she cooks Soggies on Wednesday?

(b) What is the probability that Wednesday's cereal is burned?

†17. Under the conditions of Exercise 14, the probability of burned cereal on Monday is .1, on Tuesday .13 (work this out for yourself). In Exercise 15(b) you found that the probability that Wednesday's cereal is burned is .145. It appears that, as the days pass, the probability that the cereal will be burned increases. Find P(cereal burned on Thursday).

8-3. Bayes' Formula

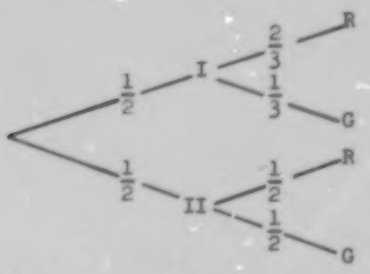
We begin this section by returning to the urn problem of the beginning of Section 8-1.

Urn I: 2 red, 1 green marble.

Urn II: 1 red, 1 green marble.

An urn is selected at random and a marble is drawn at random from that urn.

Our tree diagram is:



$\frac{2}{3}, \frac{1}{2}$	1. $P(R\|I) =$ ______ and $P(R\|II) =$ ______.
$\frac{1}{3}(= \frac{1}{2} \cdot \frac{2}{3})$	2. $P(I \cap R) = P(I) \cdot P(R\|I)$ $=$ ______.
$P(II) \cdot P(R\|II)$	3. $P(II \cap R) = \underline{P(\quad) \cdot P(\quad)}$
$\frac{1}{4}(= \frac{1}{2} \cdot \frac{1}{2})$	$=$ ______.
$P(II \cap R)$	4. $P(R) = P(I \cap R) + \underline{P(\quad)}$
$\frac{1}{4}$	$= \frac{1}{3} +$ ______
$\frac{7}{12}$	$=$ ______.

Our results (Items 1 to 4) tell us all about the probability of "red". We have $P(R|I)$, $P(R|II)$, $P(I \cap R)$, $P(II \cap R)$, and finally, $P(R)$.

Suppose now that an urn is selected and a marble drawn. We look at the marble and it is, in fact, red. At this point we are no longer interested in the probability of red. Of course, $P(R|R) = 1$.

On the other hand, we knew originally that $P(I) = \frac{1}{2}$, $P(II) = \frac{1}{2}$. Now we have new information: red is drawn. It is natural to ask: knowing that a red marble is drawn, which urn was selected? We seek, then, to determine

$$P(I|R) \quad \text{and} \quad P(II|R).$$

There is nothing new about this type of problem. We handled similar situations in Chapter 7. It may be instructive to work this particular problem through in some detail. As you suspect, we are trying to develop a general rule (Bayes' formula).

Perhaps we can guess at the answer to our urn problem.

3 5. There were ________ red marbles altogether.
(how many)

2, 1 6. Of these ______ were in Urn I, ______ in Urn II.

Knowing that a red marble is drawn does not tell us which urn was selected. It might be reasonable to argue: There were originally twice as many red marbles in Urn I; it is twice as likely that the marble came from Urn I as from Urn II.

7. Thus, we might guess

$\frac{2}{3}, \frac{1}{3}$ $P(I|R) =$ ______, $P(II|R) =$ ______.

(Surely $P(I|R) + P(II|R) = 1$. The red marble came from one of the urns!)

You have learned that guesses, which at first seem reasonable, are not always correct. (See Sections 7-5 and 8-2, for examples.) Let us see whether, in this case, our guesses are valid.

	8. We do know that $$P(I\|R) = \frac{P(I \cap R)}{\square}$$
$P(R)$	
$\frac{1}{3}$	9. Ah! We know $P(I \cap R) =$ ______ (Item 2)
$\frac{7}{12}$	and $P(R) =$ ______ (Item 4).
$\frac{\frac{1}{3}}{\frac{7}{12}}$	10. So $P(I\|R) = \dfrac{\frac{1}{3}}{\square}$
$\frac{4}{7}$	= ______
$\frac{3}{7}$	11. Similarly, $P(II\|R) =$ ______.
	Once again, our seemingly "reasonable" guess fails to be correct. On the other hand, $P(I\|R)$ is greater than $P(II\|R)$.

If you would like to follow a different but "reasonable" argument to convince yourself that $\frac{4}{7}$ and $\frac{3}{7}$ are indeed the correct values, read Items 12 to 17.

	Suppose the experiment were repeated, say, 600 times.
	On (about) 300 of the trials Urn I would be selected.
300	12. On (about) ______ of the trials Urn II would be selected.
200	13. Of the 300 selections of Urn I a red marble would be drawn (about) ______ times.
150	14. Of the 300 selections of Urn II a red marble would be drawn (about) ______ times.
200 + 150 = 350	15. That is, after 600 trials you would see a red marble (about) 200 + ______ = ______ times.

200

16. Of these 350 red marbles, how many came from Urn I? ______

$\frac{200}{350} = \frac{4}{7}$

17. What fraction of the 350 red marbles came from Urn I? $\frac{\square}{350} = \underline{\quad}$.

Now let us see if we are able to use the steps of Items 1 to 11 to reach a general procedure.

From our example:

(a) $P(I|R) = \frac{P(I \cap R)}{P(R)}$, by the conditional probability formula (1) of Section 8-1.

Look at the numerator of the fraction in (a).

(b) $P(I \cap R) = P(I) \cdot P(R|I)$, by formula (2) of Section 8-1.

Now look at the denominator of the fraction in (a).

$P(R) = P(I \cap R) + P(II \cap R)$ since a red marble must come from either I or II.

So,

(c) $P(R) = P(I) \cdot P(R|I) + P(II) \cdot P(R|II)$, again using formula (2) of Section 8-1.

Substituting (b) and (c) in (a), we have

$$P(I|R) = \frac{P(I) \cdot P(R|I)}{P(I) \cdot P(R|I) + P(II) \cdot P(R|II)}$$

This last result is Bayes' formula for the special case we have considered. You should not try to memorize this formula. (See Remark (3), page 168.)

To help you understand this rather complicated looking formula, we shall work through another problem.

Twin brothers, Ed and Jim, deliver the evening newspaper 6 nights a week. Ed delivers on 2 nights, chosen at random, and Jim on the other nights. They ride by a house on their bicycles and throw the newspaper onto the porch. The probability that Ed hits the door is $\frac{3}{5}$ and the probability that Jim hits the door is $\frac{1}{10}$. One night Mr. Jones is watching TV before dinner, when he hears a paper crash against the door. He sighs to Mrs. Jones, "It must be Ed's night with the papers." What is the probability that he is right?

$\frac{1}{3}$	18. If E is the event: Ed delivers the paper, then $P(E) =$ ______.
$\frac{2}{3}$	19. If J is the event: Jim delivers the paper, then $P(J) =$ ______.
$P(H\mid E)$	20. If H is the event: door is hit, then $\frac{3}{5} = \underline{P(\quad\mid E)}$.
$P(H\mid J)$	21. $\frac{1}{10} = \underline{P(H\mid\quad)}$.
$P(E\mid H)$	22. We are trying to find the probability that Ed delivered the paper, given that the door is hit. That is, we need $\underline{P(\ \mid\ )}$.
$\frac{P(E\cap H)}{P(H)}$	23. $P(E\mid H) = \underline{\frac{P(\qquad)}{P(\quad)}}$.
$P(E)\cdot P(H\mid E)$	24. $P(E\cap H) = \underline{P(\quad)\cdot P(H\mid E)}$.
$\frac{1}{3}\cdot\frac{3}{5}$	25. Numerically, $P(E\cap H) = \underline{\frac{1}{3}\cdot\qquad}$.
	We now know the value for the numerator of our expression $$P(E\mid H) = \frac{P(E\cap H)}{P(H)} .$$ It remains to find the value of the denominator, $P(H)$. $P(H)$ is, of course, the probability that the door is hit. It can be hit in one of two (mutually exclusive) ways. Either Ed hits the door or Jim hits the door.
$P(J\cap H)$	26. $P(H) = P(E\cap H) +$ ______________. We have found $P(E\cap H) = P(E)\cdot(P(H\mid E) = \frac{1}{3}\cdot\frac{3}{5}$.
$P(J)\cdot P(H\mid J)$ $= \frac{2}{3}\cdot\frac{1}{10}$	27. $P(J\cap H) = \underline{P(\quad)}\cdot\underline{P(\quad)} = \frac{2}{3}\ \underline{\qquad}$.

We are now able to write,

$$P(H) = \frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{10} \ .$$

Finally,

28. $$P(E|H) = \frac{\frac{1}{3} \cdot \frac{3}{5}}{\frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{10}} = \frac{\frac{\square}{15}}{\frac{\square}{15}} = \underline{\qquad}$$

Since the door was hit, the probability that Ed delivered the paper is $\frac{3}{4}$.

$$\frac{\frac{3}{15}}{\frac{4}{15}} = \frac{3}{4}$$

Throughout Items 18-28 we left the multiplication of the fractions "indicated" until the last step. We did this deliberately so that you could identify the various fractions in Item 28.

$$P(E|H) = \frac{\overset{P(E)}{\overset{\downarrow}{\frac{1}{3}}} \cdot \overset{P(H|E)}{\overset{\downarrow}{\frac{3}{5}}}}{\underset{P(E)}{\underset{\uparrow}{\frac{1}{3}}} \cdot \underset{P(H|E)}{\underset{\uparrow}{\frac{3}{5}}} + \underset{P(J)}{\underset{\uparrow}{\frac{2}{3}}} \cdot \underset{P(H|J)}{\underset{\uparrow}{\frac{1}{10}}}}$$

Once again,

$$P(E|H) = \frac{P(E) \cdot P(H|E)}{P(E) \cdot P(H|E) + P(J) \cdot P(H|J)}$$

Exercises. (Answers on page 323.)

1. Write a similar formula for $P(J|H)$. Substitute the appropriate numerical values for the various probabilities. Verify that $P(J|H) = \frac{1}{4}$.

2. We have worked this problem without reference to a tree diagram. Make a tree diagram, labeling the pieces and branches with the appropriate probabilities.

The two examples of this section dealt with situations in which a certain event ("red" for the urn problem; "hit" for the newspaper boy problem) is the union of two mutually exclusive events. We had

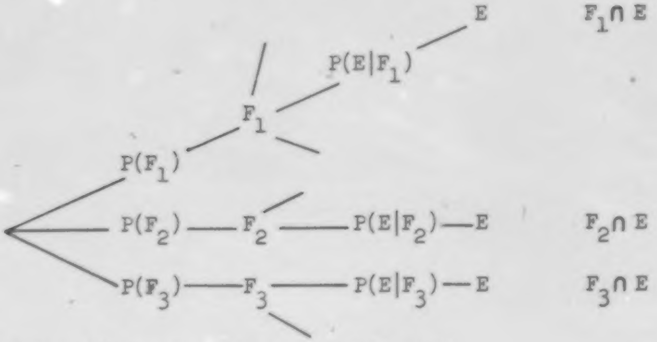
$$R = (I \cap R) \cup (II \cap R)$$

$$\text{and} \quad H = (E \cap H) \cup (J \cap H) \ .$$

Suppose some event, E, is the union of three mutually exclusive events.

$$E = (F_1 \cap E) \cup (F_2 \cap E) \cup (F_3 \cap E)$$

A portion of the accompanying tree diagram would look like the following. (The short lines indicate other branches that do not involve E.)



$$P(E) = P(F_1 \cap E) + P(F_2 \cap E) + P(F_3 \cap E)$$

$$P(F_1|E) = \frac{P(F_1 \cap E)}{P(E)}$$

$$= \frac{P(F_1) \cdot P(E|F_1)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + P(F_3) \cdot P(E|F_3)}$$

Remarks:

(1) To obtain similar formulas for $P(F_2|E)$, $P(F_3|E)$ it is only necessary to modify the numerator accordingly.

(2) If E is the union of more than three mutually exclusive events, it is clear how Bayes' formula is extended.

(3) There is no necessity to memorize Bayes' formula. It is simply a restatement, in different form, of the conditional probability

formula of Section 7-3 . The numerator of Bayes' formula reminds us that $P(F_1 \cap E) = P(F_1) \cdot P(E|F_1)$. The denominator is just another expression for $P(E)$. Any problem in conditional probability in this course may be solved by using the methods of Chapter 7. Of course, tree diagrams are often helpful.

8-4. A Further Example: Exercises.

Example:

A factory has 4 machines producing axe handles. Machine I produces 30 percent of the output; machine II produces 25 percent; machine III produces 20 percent; and machine IV produces the rest. Defective handles produced by each machine are 5 percent, 4 percent, 3 percent, and 2 percent, respectively. A handle chosen at random from the total output of the factory is examined and found to be defective. What is the probability that it was made by machine I, II, III, IV ?

	1. Originally we knew that if a handle is chosen at random, the probability that it came from machine I
.30	is $P(I) =$ ______.
	Now we have additional information. The handle is defective. We are interested in $P(I\|D)$. We use D for the event "The chosen handle is defective".
	2. Bayes' formula applied to this problem is:
$P(I)\cdot P(D\|I)$	$P(I\|D)=\dfrac{P(I)\cdot\underline{\qquad\qquad}}{P(I)\cdot P(D\|I)+P(II)\cdot P(D\|II)+P(III)\cdot P(D\|III)+P(IV)\cdot P(D\|IV)}$
	Our given information is sufficient for us to know all the individual probabilities on the right hand side of the last equation. For example:
.25	3. $P(II) =$ ______,
.04	4. $P(D\|II) =$ ______.
.015	5. Substituting the appropriate numerical values, we find that the numerator = $\underset{\text{(decimal)}}{\underline{\qquad\qquad}}$.
.036	6. The denominator becomes $\underset{\text{(decimal)}}{\underline{\qquad\qquad}}$.

$\frac{5}{12}$ 7. $P(I|D) = \frac{.015}{.036} = \frac{\square}{12}$.

The probability that the defective handle came from machine I is $\frac{5}{12}$.

If you had trouble, compare your work with:

$$P(I|D) = \frac{(.3)(.05)}{(.3)(.05) + (.25)(.04) + (.2)(.03) + (.25)(.02)}$$

.036 8. The original probability that a handle chosen at random is defective (P(D)) is ______.

P(D) is, of course, the denominator of our fraction.

You should now be able to compute the probabilities that the defective handle came from machines II, III, IV. Remember that you will use the same denominator, P(D), for all of your calculations.

$\frac{5}{18}$ 9. $P(II|D) =$ ______.

$\frac{1}{6}$ 10. $P(III|D) =$ ______.

$\frac{5}{36}$ 11. $P(IV|D) =$ ______.

1 12. Check: $\frac{5}{12} + \frac{5}{18} + \frac{1}{6} + \frac{5}{36} =$ ______.

Exercises:

1. In a two-year college 60 percent of the students are freshmen, 40 percent sophomores. Of the freshmen, 70 percent are boys. Of the sophomores, 80 percent are boys. A student is chosen at random. Find the probability that the student is:

   (a) a girl.

   (b) a freshmen, given that a girl was chosen.

2. See Exercise 3, Section 8-2. The boy uses one machine without noticing which one. He receives 1 piece of gum. What is the probability that he used machine A ?

3. See Exercise 8, Section 8-2. We are told that Kate wins. What is the probability that she won the first toss?

4. See Exercise 1, Section 8-2. What is the probability that Urn II was selected, given that the marble drawn is numbered "2" ?

5. For each of the urn problems of Exercises 4, 5 and 12 of Section 8-2, find the probability that the first marble drawn is red, given that the second is red. (Each time we start with an urn containing 6 red and 4 blue marbles. In Exercise 4 we "replace". In Exercise 5 we do not replace. In Exercise 12 we replace and add two of the first color drawn. Before doing the calculations, you should be able to judge for which of these situations the conditional probability is the least and for which it is the greatest.)

6. Use Bayes' formula to answer Exercise 8(f), Section 7-4.

7. Use Bayes' formula to answer Exercise 9, Section 7-4.

Chapter 9

BERNOULLI TRIALS

9-1. An Experiment.

A game has the following rules. Throw a die three times. You win one point each time you throw a 5 or 6. Otherwise you score nothing. Play the game 10 times. Record your results in a table as shown.

First Throw	Second Throw	Third Throw	Score
6	3	5	2

1. How many times was your score 3 ? 2 ? 1 ? 0 ? Record as shown on a table of frequencies.

Score	0	1	2	3
Number of times				

2. Do your results seem reasonable? For example, did you expect that the number of games in which your score was 3 would be about the same as the number in which your score was 0 ?

3. What was your average score per game?

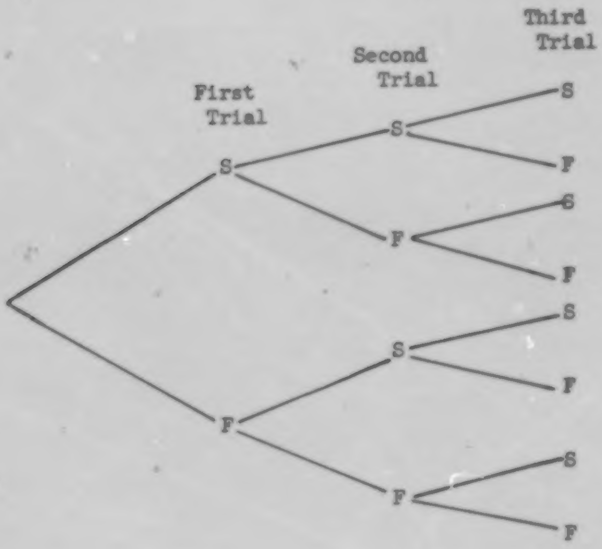
4. Suppose you have a spinner colored red and blue, but no die. You would like to use the spinner for experimenting with the game. Could you? What must be true of the spinner to permit this?

This experiment is discussed on page 306.

9-2. Introduction.

In a certain dice game a player throws one die three times. He scores one point each time the die shows 5 or 6. What is the probability that his total score is exactly 2 ? (Notice that this is the game described in the experiment of Section 9-1.)

As usual, we first construct an appropriate set of outcomes. A tree helps us to visualize them. We are interested only in whether or not a throw wins a point. Hence, we let S stand for success (winning a point) and F for failure.



We may list the possible outcomes as:

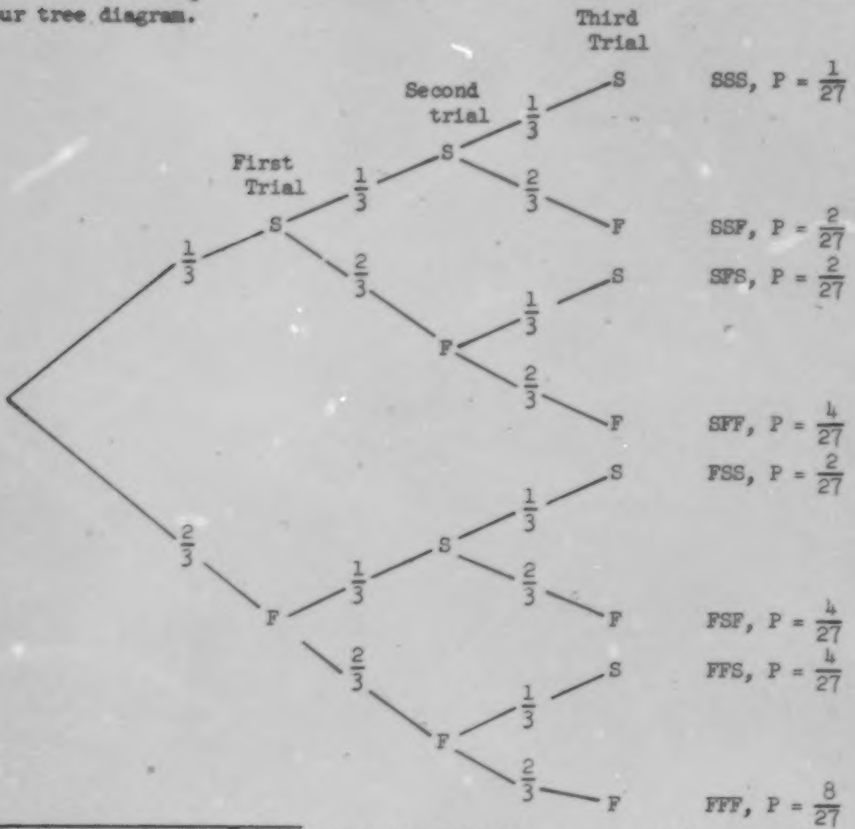
SSS, SSF, SFS, SFF, FSS, FSF, FFS, FFF.

SFS, for example, is a shorthand for "success on first throw and failure on second and success on third".

SFS	1. The player throws in succession 6, 3, 5. To which of the above outcomes does this correspond? ______
no	2. Are success and failure on the first throw equally likely? ______

$\frac{1}{3}$ $\frac{2}{3}$	3. Let P(S) be the probability of a success on any given throw and P(F) be the probability of failure. P(S) = ______. S = {5,6} P(F) = ______. F = {1,2,3,4}
P(S)	4. Since the individual throws are independent, P(SFS) = P(S) · P(F) · ______.
$\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{27}$	5. P(SFS) = ______ · ______ · ______ = ______.
FSS	6. We want to find the probability that the total score is 2. The outcomes are SSF, SFS and ______.
$\frac{2}{9}(= \frac{2}{27} + \frac{2}{27} + \frac{2}{27})$ (or $3 \cdot \frac{2}{27}$)	7. Since each of these outcomes has probability $\frac{2}{27}$, P(score of 2) = ______.

The probabilities of S and of F on a single throw can be indicated on our tree diagram.*


Third Trial
Second trial
First Trial
S
F
1/3
2/3
SSS, P = 1/27
SSF, P = 2/27
SFS, P = 2/27
SFF, P = 4/27
FSS, P = 2/27
FSF, P = 4/27
FFS, P = 4/27
FFF, P = 8/27

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* Readers of Chapter 8 will recognize this tree as a special case of the more general situation discussed in Section 8-1.

Items 8 to 17 refer to the tree diagram.

$\frac{1}{3}$	8. The probability of obtaining S on the first throw is ______.
$\frac{2}{3}$	9. The probability of obtaining F on the second throw is ______.
$\frac{2}{3}$	10. The probability of obtaining F on the third throw is ______.
$\frac{4}{27}(= \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3})$	11. The probability of the sequence SFF is ______. Notice that this can be found by multiplying the probabilities along the branch.
	Look at any branch containing one S and two F's. The probability of such a branch is the product of three factors.
once	12. In this probability, $\frac{1}{3}$ appears as a factor ______ (how many times).
twice	$\frac{2}{3}$ appears as a factor ______ (how many times).
	In each case, we can find the probability for a branch by multiplying the probabilities for the pieces along that branch.
FFS	13. The event "exactly one S in three trials" contains the outcomes SFF, FSF, and ______.
$\frac{4}{27}$	14. Each of these outcomes has probability ______.
$\frac{4}{27} + \frac{4}{27} + \frac{4}{27}$	15. P(exactly one S in three trials) = $\frac{4}{27} + \frac{4}{27} +$ ______.
$3 \cdot \frac{4}{27}$ $(= \frac{4}{9})$	16. This last is more simply written as P(exactly one S in three trials) = $3 \cdot$ ______ = ______.
$3 \cdot \frac{2}{27}$	17. In a similar way, P(exactly two S's in three trials) = $3 \cdot$ ______ $= \frac{2}{9}$.

Let us re-examine Item 17.

$$P(\text{exactly two } S\text{'s in three trials}) = 3 \cdot \frac{2}{27}.$$

Where did the $\frac{2}{27}$ come from? For each appropriate branch we multiplied the fractions $\frac{1}{3}, \frac{1}{3}, \frac{2}{3}$ in some order. It is helpful to write:

$$\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = (\frac{1}{3})^2 \cdot \frac{2}{3}.$$

$3(\frac{1}{3})^2(\frac{2}{3})$	18. The probability of exactly two S's in three trials is $\underline{3 \cdot (\quad)^2(\quad)}$.
$3(\frac{1}{3})(\frac{2}{3})^2$	19. The probability of exactly one S in three trials is $\underline{3 \cdot (\quad)(\quad)^2}$.
	You should notice that the event "exactly one S" is the same as the event "exactly two F's".
3	20. This is because we have ________ (how many) trials, each of which must yield either S or F.

The example we have been considering is typical of many problems. There are three special features of this example that need emphasis.

(1) The repeated trials of the experiment are independent.

(2) Each trial results in one of two outcomes.

(3) The probabilities of the two outcomes remain the same from trial to trial.

Trials satisfying these conditions are called Bernoulli trials, after the mathematician James Bernoulli.

Bernoulli	21. Tossing a coin 5 times forms a sequence of ______ trials.
2	22. There are ______ outcomes for each trial, H and T.

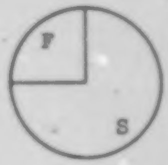
$\frac{1}{2}, \frac{1}{2}$ 23. $P(H) =$ ______ and $P(T) =$ ______.

$32 (= 2^5)$ 24. The total number of possible outcomes for the five trials is ______.

$\frac{1}{32} (= (\frac{1}{2})^5)$ 25. A typical outcome is HTTHT. What is the probability of this outcome? ______

In this case, all 32 outcomes are equally likely.

Spinning this spinner 4 times will give a sequence of Bernoulli trials.



$\frac{3}{4}, \frac{1}{4}$ 26. For each spin, $P(S) =$ ______, $P(F) =$ ______.

$16 (= 2^4)$ 27. The total number of possible outcomes for four trials is ______.

$\frac{9}{64} (= \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4}$, or $(\frac{3}{4})^2 (\frac{1}{4})^2)$ 28. A typical outcome is SFFS. What is the probability of this outcome? ______

Notice that the response to Item 28 is not $\frac{1}{16}$.

The 16 outcomes are not equally likely.

29. Which of the following does not form a sequence of Bernoulli trials?

[A] Throw a die 10 times. Record "odd" or "even".

[B] Throw a die 10 times. Record 1, 2, 3, 4, 5 or 6.

[C] Throw a die 10 times. Record 3 or "not-3".

In [B] there are 6 possible outcomes for each trial. Bernoulli trials must involve just two outcomes, so your response should have been [B].

Exercises. (Answers on page 328.)

1. Draw a tree diagram for 2 Bernoulli trials where $P(S) = .4$. Label each piece and branch with the appropriate probability. Find the probability of exactly 0, 1, 2 successes.

2. Draw a tree diagram for 3 trials of tossing a coin. (You may wish to use H, T instead of S, F.) Use the tree to find the probability of 0, 1, 2, 3 heads.

3. An urn contains 3 red, 2 blue marbles. A marble is drawn at random, its color noted, and is then replaced. Again a marble is drawn at random and its color noted. What is the probability of obtaining 0, 1, 2 red balls in the three draws?

9-3. A General Formula.

Here is a further example of Bernoulli trials.

A baseball player's batting average is .300. Assume that each "at bat" is an independent trial.

.26 1. If he comes to bat 4 times, what is the probability that he gets exactly 2 hits? ______

If you need help, complete Items 2 to 8.

We are dealing with 4 trials. Letting H stand for "hit" and N stand for "not-hit", we may list the outcomes as:

HHHH, HHHN, etc.

NHHN	2. The outcome "hit second and third times at bat and made an out other times" would be indicated ______.
16	3. There are in all ______ possible outcomes, as you can find if you list them all. (Make a tree diagram if you need to.)
6	4. Of these, the event "exactly 2 hits" contains ____ outcomes. (List them, if necessary: HHNN, HNHN, etc.)
.7	5. For each trial, the probability of H is .3 and that of N is _____. (Either he gets a hit or he doesn't. Hence, the probability of his not getting a hit is 1-.3, or .7.)
$.7 \times .3 \times .3 \times .7$, .0441	6. The probability of the outcome NHHN is $.7 \times .3 \times \_\_ \times \_\_$, or ______, which is approximately .044. We may write the product $.7 \times .3 \times .3 \times .7$ as $(.3)^2(.7)^2$.
$(.3)^2(.7)^2$	7. Similarly, the probability of each outcome with exactly two hits is $(.3)^2(\ \ )^2$.
$6 \cdot (.3)^2(.7)^2$ .26	8. Since 6 outcomes in all contain H exactly twice, the probability that the batter will make exactly two hits is approximately $\_\_ \cdot (\ \ )^2(\ \ )^2$, or approximately ______.

This problem, like the examples in the last section, deals with repeated trials of an experiment such that:

(1) the repeated trials are independent;

(2) each trial results in one of two outcomes; and

(3) the probabilities of the two outcomes remain the same from trial to trial.

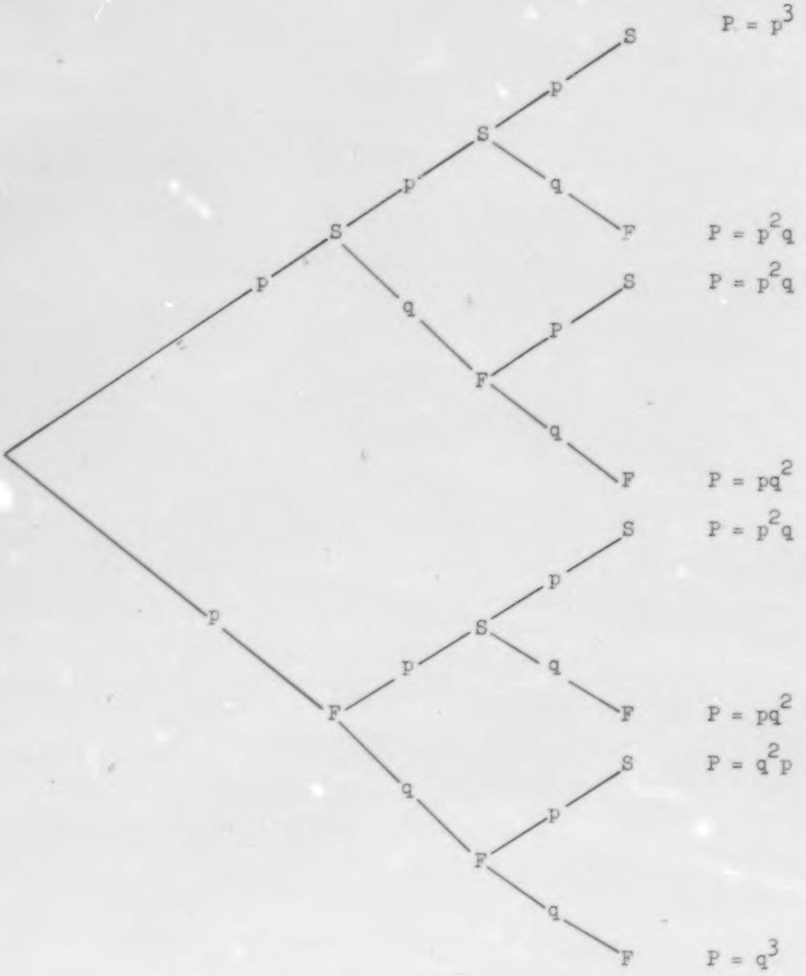
We can state a general formula that can be used in such problems. That is, we can state a general formula for the probability of obtaining exactly k successes in n Bernoulli trials.

$n - k$	9. If there are exactly k successes in n trials, there are ______ failures.
	Let p be the probability of success on any single trial.
$q = 1 - p$	10. If q is the probability of a failure, then $\underline{q = 1 - \quad\quad}$.
$p + q = 1$	11. This is true because $\underline{p + q = \quad\quad}$.
$p^k q^{n-k}$	12. The probability of each single outcome containing k successes and $n - k$ failures is $\underline{p^{\square}q^{\square}}$. Hence, the probability of exactly k successes is: $\left(\begin{matrix}\text{number of possible outcomes}\\ \text{with exactly } k \text{ successes}\end{matrix}\right) \cdot p^k \cdot q^{n-k}$.

Test your understanding by comparing with our result in Item 8.

4	13. In it, $n =$ ______,
2	$k =$ ______,
.3	$p =$ ______,
.7	$q =$ ______,
6	number of possible outcomes with exactly k successes = ______.
2	14. In this case, $n - k =$ ______.
	We found: probability of exactly k successes is: $6 \cdot (.3)^2(.7)^2$

If we had 3 Bernoulli trials with $P(S) = p$, $P(F) = 1 - p = q$, our tree would look like this:


S
S
S
F
F
S
F
F
S
S
S
F
F
S
F
F
p
q
P = p^3
P = p^2q
P = p^2q
P = pq^2
P = p^2q
P = pq^2
P = q^2p
P = q^3

p^2q	15. For 3 trials, the probability for each branch containing 2 S's and 1 F is ______.
3	16. There are ______ (how many) such branches.
$3p^2q$	17. Hence, P(exactly 2 S's) = ______ $\cdot p^2q$.
$p^k \cdot q^{n-k}$	18. For n trials, the probability for each branch containing k S's and n-k F's is ______.

Hence, the probability of exactly k successes in n Bernoulli trials is:

$$\begin{pmatrix}\text{number of possible outcomes}\\ \text{with exactly } k \text{ successes}\end{pmatrix} \cdot p^k \cdot q^{n-k} .$$

Let us examine the product $p^k \cdot q^{n-k}$. (q, remember, is $1 - p$.)

19. If $n = 10$ and $k = 3$, then

$$p^k \cdot q^{n-k} = \underline{p^3 \cdot q^{\square}}\ .$$

$p^3 \cdot q^7$

20. If $n = 10$, is it possible to have $k = 12$? ________

No

k is the number of "successes" in n trials.
We can't have more successes than we have trials!

21. If $n = 10$, what is the largest value k can have? ________

10

22. If $n = 10$, what is the smallest value k can have? ________

0

For 10 trials we may have either 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 successes.

For 2 successes in 3 trials, the product $p^2 \cdot q^1$ is understandable, since $q^1 = q$. What if we are concerned about 3 successes in 3 trials? Our product becomes p^3q^0. What meaning should be attached to q^0 ?

Let us recall an example.

In Section 9-2 we considered a game where P(success on any trial) $= \frac{1}{3}$.

23. We saw that $P(SSS) = \underline{(\quad)}^3$.

$(\frac{1}{3})^3$

24. If our formula is to hold for this case, then

$$(\tfrac{1}{3})^3(\tfrac{2}{3})^0 \text{ must equal } \underline{(\quad)}^3 .$$

$(\frac{1}{3})^3$

But $(\frac{1}{3})^3$ times "something" can equal $(\frac{1}{3})^3$ only if the "something" is equal to 1.

25. $(\frac{1}{3})^3 y = (\frac{1}{3})^3$ only if $y =$ ______.

1

Hence, $(\frac{2}{3})^0$ must be 1 if our formula is to hold.

We decide, therefore, to define q^0 as 1. ($q \neq 0$.)

p^0q^n	26. What if $k = 0$? Then $p^k \cdot q^{n-k}$ becomes $p^{\square}q^{\square}$.
1	27. If you think a moment, you will see that we define p^0 as ______.

When you study algebra, you will find that the definition of p^0, q^0 given here is consistent with other mathematical usage.

We have seen:

Probability of exactly k successes in n Bernoulli trials;

$$\begin{pmatrix}\text{number of possible outcomes}\\ \text{with exactly } k \text{ successes}\end{pmatrix} \cdot p^k \cdot q^{n-k} .$$

It is convenient to have a symbol for the first factor in the formula. We will write $\binom{n}{k}$ for the number of possible outcomes with exactly k successes in n trials.

1; 3	28. $\binom{3}{1}$ means the number of possible outcomes with exactly ______ successes in ______ trials.
3	29. $\binom{3}{1}$ = ______. (If you weren't sure, look back at the first tree in Section 9-2.)
2; 4	30. $\binom{4}{2}$ means the number of possible outcomes with exactly ______ successes in ______ trials.
6	31. $\binom{4}{2}$ = ______. (Look back at Item 4 in this section if you weren't sure.)
1	32. $\binom{4}{0}$ = ______.
1 FFFF	33. This is true because there is only ______ outcome with no successes in 4 trials. This outcome is ____.
1	34. $\binom{4}{4}$ = ______. (The only outcome with 4 successes is SSSS.)

Notice, once again, that in the symbol $\binom{n}{k}$, k can be any of the numbers $0, 1, 2, \ldots, n$. It cannot exceed n; there can't be more successes than trials.

Our result for the probability of exactly k successes in n Bernoulli trials becomes, using our new symbol:

$$\binom{n}{k}p^k q^{n-k} .$$

In solving problems, you may prefer to think about the tree, without using the formula. This is always possible.

In the next example, the number of trials is large. We shall not carry out the necessary calculations.

A student takes a multiple-choice test in which each question has 3 choices. There are 20 questions. Since he has not studied at all, he decides simply to pick an answer at random for each question. What is the probability that he will get exactly 16 correct?

20 35. This is a situation with ______ trials.

2^{20} 36. There are 2 possible outcomes for each trial, hence $(\quad)^{20}$, possible outcomes for the 20 trials.

$\frac{1}{3}$ 37. On each trial the probability of success (getting the right answer) is ______, while the probability of
$\frac{2}{3}$ failure is ______.

4 38. He gets exactly 16 correct if he misses exactly ______ questions.

We can apply our formula to find the probability that he will get exactly 16 correct.

20

16

4

$\frac{1}{3}$

$\frac{2}{3}$

39. Here n = ______,

k = ______,

n-k = ______,

p = ______,

q = ______.

Probability $= \binom{20}{16}(\frac{1}{3})^{16}(\frac{2}{3})^{4}$.

In order to compute the probability, we are faced with the problem of finding $\binom{20}{16}$--that is, of counting how many outcomes (out of 2^{20} or 1,048,576) lead to exactly 16 successes. (It turns out that there are 4845.) In Section 9-4 we shall concentrate on this problem of counting the number of outcomes with exactly k successes on n trials.

The other difficulty, that of handling the arithmetic involved in $4845 \cdot (\frac{1}{3})^{16} \cdot (\frac{2}{3})^{4}$, may be overcome by use of a calculator or by computational methods that you will learn later in your study of mathematics. There are also tables which eliminate the need for much computation.

Warning! $\binom{n}{k}$ is a symbol for the number of ways certain outcomes occur. Hence, $\binom{n}{k}$ is a counting number, not a fraction.

Exercises (Answers on page 330.)

1. In Items 1 to 8 we found that for a baseball player with a batting average of .300, P(exactly 2 hits in 4 at bats) = .26. Complete this example by finding the probability of exactly 0, 1, 3, 4 hits in 4 at bats. You may list the outcomes or use a tree diagram to help you in counting.

   What is the probability that he makes at least 2 hits in 4 at bats?

2. Five dice are thrown. What is the probability that exactly one of them shows a 3? (Leave answer in terms of powers.)

3. (a) Think about tossing 4 coins. What is the probability of no heads; of exactly one head; of exactly 2 heads; of exactly 3 heads; of exactly 4 heads?

   (b) Try really tossing 4 coins several times. Record the number of times you get no heads; one head; etc. Compare with the results you have computed.

   (c) What is the most likely number of heads when 4 coins are tossed? What is the probability of this result?

4. Ten dice are thrown. What is the probability that at least one shows a 6 ? (Hint: You might find the probability of one 6, two 6's, etc., and add. It would be simpler, however, to begin by finding the probability of no 6's. Leave answer in terms of powers.)

9-4. The Pascal Triangle

We have seen that, in a sequence of Bernoulli trials, the probability of exactly k successes in n trials is found by multiplying $p^k q^{n-k}$ by the number of outcomes with exactly k successes. (Again, of course, p represents the probability of a success on any one of the n independent trials.)

Our next task, then, is to discover a convenient method of counting those outcomes with exactly k successes. That is, we wish to find a method of finding the value of $\binom{n}{k}$.

First of all, we should recall how many possible outcomes there are altogether.

2 — 1. For 1 Bernoulli trial there are ______ (how many) possible outcomes, S and F.

2^2, or 4 — 2. For 2 Bernoulli trials there are ______ (how many) possible outcomes: SS, SF, ______, and ______.

2^3, or 8

3. For 3 Bernoulli trials there are ________ (how many) possible outcomes.

2^n

4. In general, for n Bernoulli trials there are $2^{\square}$ possible outcomes.

For a single Bernoulli trial there are 2 outcomes. One outcome yields 1 success, the other yields 0 successes: $\binom{1}{1} = 1$, $\binom{1}{0} = 1$.

In the case of 2 Bernoulli trials we have, as the set of possible outcomes, {SS, SF, FS, FF}. We notice that there are 1, 2, 1 outcomes which yield, respectively, 2, 1, 0 successes. That is:

$$\binom{2}{2} = 1, \quad \binom{2}{1} = 2, \quad \binom{2}{0} = 1.$$

5. Develop a similar statement yourself for the case of 3 Bernoulli trials. Refer to the tree in Section 9-3, if necessary. Although it involves a bit more counting, try to extend this idea to 4 Bernoulli trials.

Find $\binom{4}{4}$, $\binom{4}{3}$, etc.

Compare your answer with the lists below.

						Total Outcomes
1 trial	$\binom{1}{1}$ 1	$\binom{1}{0}$ 1				2
2 trials	$\binom{2}{2}$ 1	$\binom{2}{1}$ 2	$\binom{2}{0}$ 1			4
3 trials	$\binom{3}{3}$ 1	$\binom{3}{2}$ 3	$\binom{3}{1}$ 3	$\binom{3}{0}$ 1		8
4 trials	$\binom{4}{4}$ 1	$\binom{4}{3}$ 4	$\binom{4}{2}$ 6	$\binom{4}{1}$ 4	$\binom{4}{0}$ 1	16

You may have seen this pattern before:

1 trial 1 1
2 trials 1 2 1
3 trials 1 3 3 1
4 trials 1 4 6 4 1

Can you guess how the next row (for 5 trials) will look? Even if you are able to guess, it is helpful to read the discussion that follows. We are going to build up the pattern for 5 trials from the pattern for 4 trials.

16	6. There are altogether ________ (how many) outcomes for 4 trials.
32	7. There are altogether ________ (how many) outcomes for 5 trials.
2	8. Every 4-trial outcome (such as SSFS) leads to ________ (how many) 5-trial outcomes. (Think about the tree diagram.)
S F	9. In particular, SSFS leads to SSFS___ and to SSFS___.
SFFS	10. The 5-trial outcome SFFSS is obtained by attaching S to the 4-trial outcome ______.

Let us find $\binom{5}{3}$ by using what we already know about $\binom{4}{3}$ and $\binom{4}{2}$. Think about the tree diagram for 5 trials. To obtain a branch (of the 5-trial tree) with 3 successes, we may proceed in either of two ways:

(a) attach an F to a 4-trial branch having 3 S's; or
(b) attach an S to a 4-trial branch having 2 S's.

4 F	11. We have ______ 4-trial outcomes with exactly 3 successes. By attaching ______ to any one of them, we get a 5-trial outcome with exactly 3 successes. (Example: The 4-trial outcome SSSF becomes SSSFF.)
6 S	12. We have ______ 4-trial outcomes with exactly 2 successes. By attaching ______ to any one of them we get a 5-trial outcome with 3 successes. (Example: SSFF becomes SSFFS.)

no	13. Are there other 4-trial outcomes that can be changed to 5-trial outcomes with three successes by attaching an S or F ? ______
6 + 4; 10	14. Thus we have found all the 5-trial outcomes with 3 successes. There are ___+___ , or ______, of them.

Now you should be able to write the complete pattern for the 5-trial outcomes.

SSSSS	15. The 5-trial pattern should begin with 1, because the only "5 success" outcome for 5 trials is SS______.
5	16. The next entry should be the number of ways of getting 4 successes. There are ______ such ways. You could see this by noting that there is just one failure for this outcome, and it can be on the first, second, third, fourth, or fifth trial.
F S	17. You could also think: I can get 4 successes by attaching ______ to SSSS. I can also get 4 successes by attaching ______ to each of SSSF, SSFS, SFSS, FSSS.
1 + 4	18. Hence, for 5 trials the total number of outcomes with 4 successes is: 1 +______, or 5.
	We have already found that $\binom{5}{3} = 4 + 6 = 10$.

By now you should see what is going on. Our results thus far can be summed up as follows:

4 trials		$\binom{4}{4}$ 1	$\binom{4}{3}$ 4	$\binom{4}{2}$ 6	$\binom{4}{1}$ 4	$\binom{4}{0}$ 1
5 trials	$\binom{5}{5}$ 1	$\binom{5}{4}$ 1 + 4, or 5	$\binom{5}{3}$ 4 + 6, or 10			

As we have seen in the 5-trial pattern, each number after the first is the sum of a pair of adjacent numbers in the 4-trial pattern. It turns out that all but the last of the numbers in the 5-trial row can be found in the same way. You should verify, by the reasoning we have used above, that these entries are correct. Check your results with:

$$\begin{array}{lcccccccccccc}
\text{4 trials} & & & 1 & & 4 & & 6 & & 4 & & 1 & \\
\text{5 trials} & 1 & & \underbrace{}_{5} & & \underbrace{}_{10} & & \underbrace{}_{10} & & \underbrace{}_{5} & & & 1
\end{array}$$

Notice that the last entry in each row is 1. In each case there is only one outcome with no successes.

We have written the patterns for 1, 2, 3, 4 and 5 trials. You should see by now how to write the row corresponding to 6 trials, and then how to go on to the 7-trial row, etc.

> 19. Write the 10 rows for 1, 2, 3, ... 10 Bernoulli trials. Check with the answer below.

Table of $\binom{n}{k}$

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

This table is part of the array known as the Pascal triangle. (It could be extended to any required number of rows.) It is named for the seventeenth century French mathematician Blaise Pascal. To make it look more like a triangle we could insert an additional "1" at the top. (We would call this the 0 row, associated with no Bernoulli trials.)

1, 27	20. What are the first two entries of the 27th row of the Pascal triangle? ______, ______.
43, 1	21. What are the last two entries of the 43rd row? ______, ______.
6 19	22. How many numbers are there in the 5th row? ______. How many are there in the 18th row? ______.
21	23. 7 coins are tossed. For how many outcomes are there exactly 5 heads? ______. If you had trouble, complete the box. If not, omit Items 24 to 26. Notice that tossing 7 coins can be considered as 7 Bernoulli trials.
7 21	24. The row of the Pascal triangle corresponding to 7 coins is: 1 7 21 35 35 21 7 1. From this row we read: There is 1 outcome with heads 7 times. There are ______ outcomes with heads 6 times. There are ______ outcomes with heads 5 times.
35	25. 7 coins are tossed. For how many outcomes are there exactly 4 heads? ______
21 + 35; 56	26. Now think of tossing 8 coins. For how many outcomes are there exactly 5 heads? 21 + ______, or ______.
	(Think again about the reasoning we used earlier, and then check with the Pascal triangle.)

Read the numbers in a row of the Pascal triangle from left to right. Now, read the same row from right to left. Did you say the same numbers in the same order both times? You did, of course. You will probably see why this must be the case if you think of how a Bernoulli tree is constructed. This symmetry is discussed again in Section 9-7.

In order to use the Pascal triangle more readily, it is convenient to arrange it in the form of the following table.

Table of $\binom{n}{k}$ (from Pascal triangle)

n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1														
2	1	2	1													
3	1	3	3	1												
4	1	4	6	4	1											
5	1	5	10	10	5	1										
6	1	6	15	20	15	6	1									
7	1	7	21	35	35	21	7	1								
8	1	8	28	56	70	56	28	8	1							
9	1	9	36	84	126	126	84	36	9	1						
10	1	10	45	120	210	252	210	120	45	10	1					
11	1	11	55	165	330	462	462	330	165	55	11	1				
12	1	12	66	220	495	792	924	792	495	220	66	12	1			
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1		
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1	
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

We have a spinner, colored red and blue. Suppose we think of spinning 8 times. There are $2^8 = 256$ possible outcomes. How many of these will show exactly 4 red?

27. Look at the 8th row. Go across it to the column with 4 at the top. The number we read is ______.

10 2	28. We spin 5 times. The number of outcomes with exactly 3 reds is ______. This is also the number of outcomes for 5 spins with exactly ______ blues.

Here are some additional problems to give you practice in using the table. If you think that you do not need the practice, go on to Section 9-5.

165	29. $\binom{11}{3} =$ ______.
36	30. $\binom{9}{7} =$ ______.
36	31. $\binom{9}{2} =$ ______.
4	32. The number of outcomes with 8 successes in 12 trials equals the number of outcomes with ______ (how many) successes in 12 trials.
=	33. $\binom{15}{6} \underset{(=,\neq)}{\underline{\qquad}} \binom{15}{9}$.
1	34. $\binom{17}{17} = \binom{17}{0} =$ ______.
	Although our table only extends to the 15th row, you should be able to complete Items 35-38.
$\binom{20}{0} = 1$	35. $\binom{20}{20} = \binom{20}{\square} =$ ______.
20	36. $\binom{20}{1} = \binom{20}{19} =$ ______.
1000	37. $\binom{1000}{1} =$ ______.
1000	38. $\binom{1000}{999} =$ ______.

9-5. The Binomial Distribution

We are now ready to return to our formula:

$$P(\text{exactly } k \text{ successes in } n \text{ Bernoulli trials}) = \binom{n}{k}p^k q^{n-k}$$

Let us start with some examples. Use the Pascal triangle to determine the value of $\binom{n}{k}$.

Assume that the probability of a successful launch of a certain type of satellite rocket is .8 . Assume further that successive launchings form a sequence of Bernoulli trials.

1. Four launchings are attempted during a given week. Complete the following table. Record the probabilities to 2 decimal places.

Number of Successes (k)	Probability $\binom{4}{k}(.8)^k(.2)^{4-k}$
4	
3	
2	
1	
0	

Compare your results with Table I, which follows Item 10. If you had difficulty, complete Items 2 to 10.

2. We use the formula:

$$P(k \text{ successes}) = \binom{n}{k}p^k q^{n-k}$$

.8; .2

with n = r, p = ______, q = ______.

$\binom{4}{4}p^4 q^0$

3. P(4 successes) = $\underline{\binom{4}{4}p^{\square}\ q^{\square}}$

$$= \binom{4}{4}\ (.8)^4(.2)^0 .$$

1	4. $\binom{4}{4} =$ ______.
.41	5. Hence, $P(4 \text{ successes}) = 1(.8)^4(.2)^0 \sim$.______. Remember that $(.2)^0 = 1.$)
$\binom{4}{3}(.8)^3(.2)^1$	6. Similarly, $P(3 \text{ successes}) = \underline{(\quad)\ (.8)^3(.2)^1}$ $= 4(.8)^3(.2)$
.41	$\sim$ ______.
$\binom{4}{2}(.8)^2(.2)^2$	7. $P(2 \text{ successes}) = \underline{(\quad)(\quad)^{\square}(\quad)^{\square}}$ $= 6(.8)^2(.2)^2$
.15	$\sim$ ______.
	8. $P(1 \text{ success}) = \binom{4}{1}(.8)^1(.2)^3$
.03	$\sim$ ______.
$\binom{4}{0}(.8)^0(.2)^4$	9. $P(0 \text{ successes}) = \underline{(\quad)(.8)^{\square}(.2)^4}$
.00	$\sim$ ______.
	10. Notice that the probability of no successful launchings in 4 trials is not exactly 0. In fact,
.0016	$P(0 \text{ successes}) =$.______.

Here is the completed table:

Number of Successes (k)	Probability $\binom{4}{k}(.8)^k(.2)^{4-k}$
4	.41
3	.41
2	.15
1	.03
0	.00

Table I

$(n = 4,\ p = .8)$

Remember that the probabilities in Table I are recorded to the nearest hundredth.

11. For a certain type of light bulb, the probability that it will burn for 200 hours is .8 . If 4 of these bulbs are installed in a certain room, then, after 200 hours,

.41 P(none have burned out) ~ ______ (two decimal places)
.41 P(exactly 1 has burned out) ~ ______.
.15 P(exactly 2 have burned out) ~ ______.
.03 P(exactly 3 have burned out) ~ ______.
.00 P(exactly 4 have burned out) ~ ______.

You could, of course, have obtained your responses to Item 11 by actually calculating each probability. Instead, you used Table I. Such a table is called a probability distribution. In this case it is the binomial[*] probability distribution for $n = 4$, $p = .8$. As you see, once a distribution has been determined, it may be used in a variety of situations.

We may consider other binomial distributions, for other values of n and p. For each value of n we have many binomial distributions--one for each value of p. Of course, if we know p, we also know q. Observe that the possible values of k are $0 \leq k \leq n$.

Let us take $n = 4$ and construct a table showing the binomial distribution for several different values of p.

12. Complete the following table. As you proceed, you may find ways to shorten your work.

k \ p	.2	$\frac{1}{3}$	.4	$\frac{1}{2}$	.6	$\frac{2}{3}$	.8
4		.01		.06			.41
3		.10		.25			.41
2		.30		.38			.15
1		.40		.25			.03
0		.20		.06			.00

Compare your results with Table II, which follows Item 18. If you had difficulty, or if you did not discover the short-cuts, read Items 13 to 18.

---

[*] A binomial distribution, as we have seen, arises in connection with a sequence of Bernoulli trials. The name "binomial" is explained in Section 9-6.

You need not do any computation at all in order to complete the column for $p = .2$.

13. Consider $k = 3$, $p = .2$. We need:

$\binom{4}{3}(.2)^3(.8)^1$

$$\underline{\binom{4}{3}(\quad)^3(\quad)^1} .$$

$\binom{4}{1}$

14. But, $\binom{4}{3} = \binom{4}{\square}$ and $(.2)^3(.8)^1 = (.8)^1(.2)^3$. Therefore,

$$\binom{4}{3}(.2)^3(.8)^1 = \binom{4}{1}(.8)^1(.2)^3 .$$

.8

15. The right-hand side of the last sentence is the entry for $k = 1$, $p =$ ______.

.03

16. This entry is ______ (decimal).

Similarly, you can immediately complete the column for $p = \frac{2}{3}$.

$(.4)^0$

17. The entry for $k = 4$, $p = .6$ is found as follows:

$$\binom{4}{4}(.6)^4\underline{(\quad)}^{\square} \sim .13 .$$

0

18. .13 is also the entry for $k =$ ______, $p = .4$.

Thus, once the column for $p = .6$ is completed, we may immediately write down the column for $p = .4$.

k \ p	.2	$\frac{1}{3}$	.4	$\frac{1}{2}$	.6	$\frac{2}{3}$	.8
4	.00	.01	.03	.06	.13	.20	.41
3	.03	.10	.15	.25	.35	.40	.41
2	.15	.30	.35	.38	.35	.30	.15
1	.41	.40	.35	.25	.15	.10	.03
0	.41	.20	.13	.06	.03	.01	.00

Table II

Binomial Distribution, $n = 4$

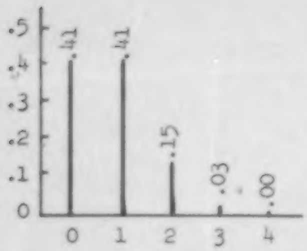
It is worth a few moments to study Table II. You should notice several general patterns.

up	19. We have already pointed out that if we read down the column for $p = \frac{1}{3}$, we find the same entries as if we read ______ the column for $p = \frac{2}{3}$.
increase	20. As we read across the row for $k = 4$, the entries ______ (increase, decrease) in value.
k = 1	21. Reading the row for $k = 3$ from right to left gives the same sequence of numbers as reading the row for $k =$ ____ from left to right.
1	22. If we add the entries in any column, the sum should be ______. In fact, the sum for certain columns is not 1. This is because we have recorded the probabilities only to the nearest hundredth.
0	23. Suppose we adjoined to our table a column for $p = 0$. Then the entry for $k =$ ____ would be 1, and all other entries would be 0.
k = 4	24. Similarly, a column for $p = 1$ would contain an 0 in each row, except for $k =$ ____.
$p = \frac{1}{2}$	25. Finally, only one column of Table II reads the same from top to bottom as from bottom to top. This is the column for $p =$ ____.

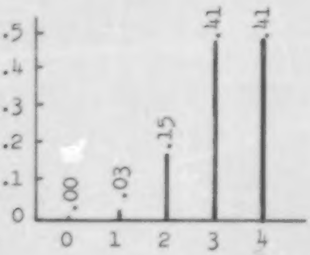
In order to visualize these relationships, it is convenient to draw bar graphs. The graphs on page 200 show the data of Table II. The vertical scale is marked with the probabilities, the horizontal scale with the values of k.

Binomial Distributions

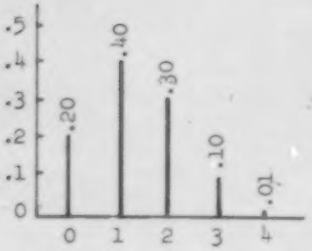
(n = 4)



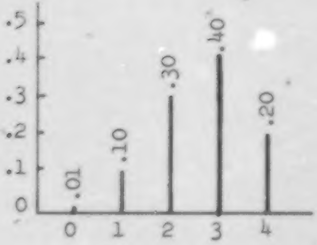
p = .2



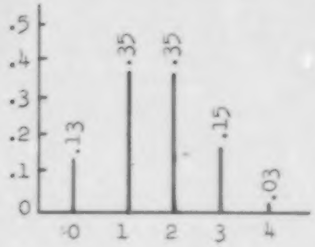
p = .8



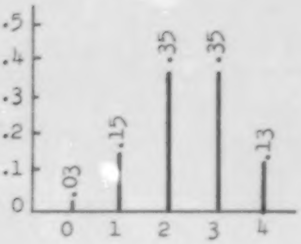
p = $\frac{1}{3}$



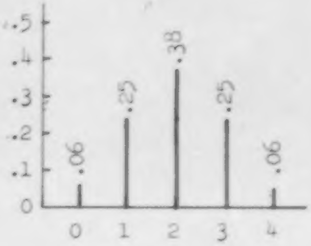
p = $\frac{2}{3}$



p = .4



p = .6



p = $\frac{1}{2}$

Table II shows the binomial distribution for a particular value of n, namely 4, and for various values of p.

It is also interesting to construct a table for a particular value of p and for various values of n. We choose $p = \frac{1}{2}$.

$q = \frac{1}{2}$

Before we begin, we observe that every column of our table will read the same from top to bottom as from bottom to top.

26. This is because with $p = \frac{1}{2}$, we also have $q =$ ______.

Binomial Distribution

$(p = \frac{1}{2})$

n k	1	2	4	5	6	8	10
0	.50	.25	.06	.03	.02	.00	.00
1	.50	.50	.25	.16	.09	.03	.01
2		.25	.38	.31	.23	.11	.04
3			.25	.31	.31	.22	.12
4			.06	.16	.23	.27	.21
5				.03	.09	.22	.25
6					.02	.11	.21
7						.03	.12
8						.00	.04
9							.01
10							.00

Table III

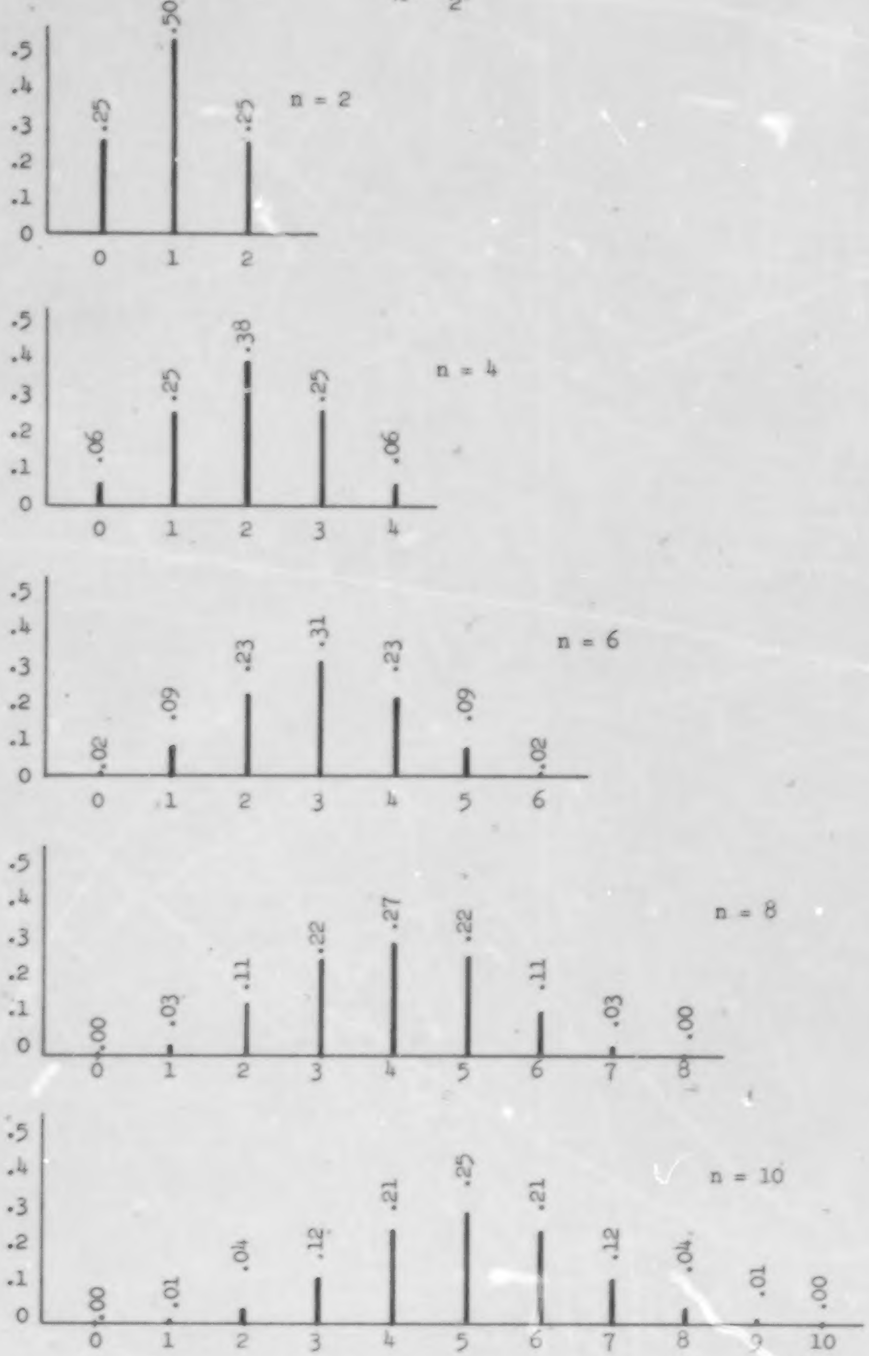
Let us not forget what this table means. We can interpret it, for example, as the probability of getting k heads when we throw n coins.

.23	27. If we throw 6 coins, the probability that exactly 2 of them are heads is, from the table, ______.
3	28. If we throw 6 coins, the number of heads most likely to appear is ______. This fits our intuitive idea, since heads and tails are equally likely on coin throws.
.31	29. However, the probability of throwing exactly 3 heads with 6 coins is not very great--only ______, which is less than $\frac{1}{3}$.
.34(= .23+.09+.02)	30. The probability of throwing more than 3 heads with 6 coins is ______. (Hint: The probability of throwing more than 3 heads is the probability of throwing 4, 5 or 6 heads.)

Again, it is instructive to display the distributions in the form of bar graphs. On page 203 are the bar graphs for $p = \frac{1}{2}$ and $n = 2, 4, 6, 8, 10$.

Binomial Distributions

$(p = \frac{1}{2})$



When you examine the bar graphs for $p = \frac{1}{2}$, you note several things about them.

First, they are all symmetrical, which by now does not surprise us.

Second, the bigger n, the more widely the bar graph spreads out. Again, this does not surprise us, because for each n the values of k are $0, 1, 2, \ldots, n$. (See Chapter 10 for more discussion of this idea.)

Third, the bar graph for each value of n is flatter than the one for the preceding value. The tallest bar is less tall when n is larger.

The tallest bar of each graph corresponds to that value of k which is "most likely".

31. Complete the table and compare with answer below.

n	Most Likely Value of k	Probability of the Most Likely Value
2	1	
4		.38
6		
8		
10		.25

The most likely value is often called the expected value. Notice how the probabilities of the expected value decrease as n increases. For more discussion of expected value, see Chapter 10.

n	Most Likely Value of k	Probability of the Most Likely Value
2	1	.50
4	2	.38
6	3	.31
8	4	.27
10	5	.25

Item 31 reveals that, for our bar graphs, the highest point is further to the right when n is large than when n is small. We sometimes say, "As n increases, the most likely value of k 'walks' to the right".

We have considered earlier the bar graph for $n = 4$, $p = \frac{1}{3}$. If we used tables to draw other bar graphs, using this value of p (that is, $\frac{1}{3}$) and $n = 6$, $n = 8$, $n = 10$, we would observe again:

> For larger values of n, the bar graph spreads out and flattens. The most likely value of k "walks" to the right.

If you have already learned about functions and their graphs, then you will understand that each of our histograms can be turned into the graph of a function. Look back at the bar graph with $n = 4$, $p = \frac{1}{3}$. We can consider the function $f(k)$, where $f(k) = \binom{4}{k}(\frac{1}{3})^k(\frac{2}{3})^{4-k}$.

$\{0, 1, 2, 3, 4\}$

32. The domain of this function is $\{\underline{\quad\quad}\}$.

If you put a red dot at the top of each line in the bar graph, this set of red dots is the graph of the function.

Exercise. (Answer on page 330.)

In your record of 100 throws of a die, you have 20 blocks of 5 throws. Count the number of even numbers in each block of 5, recording as shown:

Number of Evens	0	1	2	3	4	5
Number of times appearing						

Compare your results with the bar graph for $n = 5$, $p = \frac{1}{2}$, $q = \frac{1}{2}$. Our results are on page 330.

9-6. The Binomial Theorem

The work we have done with the Pascal triangle is very closely related to a topic of algebra. You will understand both better if you see this relation. An expression such as $x + y$ is an example of a binomial. It is the sum of two terms. We are often interested in finding powers of binomials. That is, we wish to find expressions for $(x + y)^2$, $(x + y)^3$, etc.

$(x + y)(x + y)$
$(x+y)x+(x+y)y$

1. You know that if x and y are real numbers, then

$$\begin{aligned}(x + y)^2 &= \underline{(x + y)(\qquad)}\\ &= \underline{(x + y)x + (\qquad)y}\\ &= xx + yx + xy + yy.\end{aligned}$$

(Here we are applying the distributive property.)

$2xy$

commutative

2. For most purposes, we simplify further:

$$(x + y)^2 = x^2 + \underline{\quad xy} + y^2 .$$

We know that $xy = yx$ from the _______ property of multiplication.

$(x + y)^2(x + y)$
$x^3 + 3x^2y + 3xy^2 + y^3$

3. Similarly,

$$\begin{aligned}(x + y)^3 &= \underline{(x + y)^{\square}(x + y)}\\ &= (x^2 + 2xy + y^2)x + (x^2 + 2xy + y^2)y\\ &= \underline{x^3 + 3x^2y + \qquad\qquad}.\end{aligned}$$

$x^4 + 4x^3y + 6x^2y^2$
$+ 4xy^3 + y^4$

4. Let's try one more:

$$(x + y)^4 = \underline{x^4 + \qquad\qquad}$$

(Hint: $(x + y)^4 = (x + y)^3(x + y)$).

We have:

$$\begin{aligned}(x + y)^1 &= x + y\\ (x + y)^2 &= x^2 + 2xy + y^2\\ (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

5. Look carefully for a pattern. Are you reminded of something? Even if you don't know, or if you are not sure why, you can at least guess:

$$(x + y)^5 = \_\_\_\_\_\_.$$

The coefficients in the expansion of $(x + y)^5$ can be read, it appears, from the Pascal triangle.

$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

It is not very difficult to see why. Let us go back to our work with $(x + y)^2$. Let us suppose for a moment that we did not simplify after using the distributive property. We have

$$(x + y)^2 = xx + yx + xy + yy.$$

From this we can find:

$$\begin{aligned}(x + y)^3 &= (x + y)^2(x + y)\\ &= (xx + yx + xy + yy)(x + y)\\ &= xxx + yxx + xyx + yyx + xxy + yxy + xyy + yyy.\end{aligned}$$

Notice that in

$$xx + yx + xy + yy$$

we have terms like xx, yx. The four terms are all the 2-letter "words" we can write with x and y. In finding $(x + y)^3$, we attach to each of these two-letter words first an "x" and then a "y". This leads, not surprisingly, to all the 3-letter words we can write with the letters x and y.

6. The 3-letter words with exactly two x's are:

yxx, xyx, and ______.

xxy

7. Remembering, now, the commutative property of multiplication, along with other properties of the real numbers, we see that

$$yxx + xyx + xxy = \underline{\quad x^2y\quad}.$$

A little more grouping and rewriting leads to the result:

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

$3x^2y$

Again, we could compute $(x + y)^4$ by attaching first x and then y to all the 3-letter words in x and y.

All this should remind you very much of what we did when we first developed the Pascal triangle. It should help you understand why the pattern of coefficients in $(x + y)$, $(x + y)^2$, $(x + y)^3$, etc., is given by the Pascal triangle.

In the 4th row of the Pascal triangle we see the numbers

$$1 \quad 4 \quad 6 \quad 4 \quad 1.$$

8. From this we see that if p and q are any real numbers, then we can write:

$$(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + \underline{\qquad + \qquad}.$$

$4pq^3 + p^4$

Suppose we are thinking of a baseball player whose batting average is .333, or $\frac{1}{3}$.

9. Each time he is at bat the probability of a hit is $\frac{1}{3}$, while the probability that he makes an out is ______.

$\frac{2}{3}$

Suppose that this player is at bat 4 times. We have already found the probabilities for no hits, for exactly one hit, etc.

10. The probability for exactly two hits is:

$$\underline{6(\tfrac{1}{3})^{\square}(\tfrac{2}{3})^{\square}}.$$

$6(\frac{1}{3})^2(\frac{2}{3})^2$

11. The probability for exactly one hit is ______.

$4(\frac{1}{3})(\frac{2}{3})^3$

Notice that if we compute $(\frac{1}{3} + \frac{2}{3})^4$, using the pattern shown in Item 8, we have:

12. $(\frac{1}{3} + \frac{2}{3})^4 = (\frac{1}{3})^4 + 4(\frac{1}{3})^3(\frac{2}{3}) + 6(\frac{1}{3})^2(\frac{2}{3})^2 + \underline{\qquad + \qquad}.$

$4(\frac{1}{3})(\frac{2}{3})^3 + (\frac{2}{3})^4$

The terms $(\frac{1}{3})^4$, $4(\frac{1}{3})^3(\frac{2}{3})$, etc., are simply the probabilities for exactly 4 hits, exactly 3 hits, etc.

1 13. Of course, $\frac{1}{3} + \frac{2}{3} =$ ______, and consequently,

1 $(\frac{1}{3} + \frac{2}{3})^4 =$ ______.

So our algebraic results and our knowledge of probability fit together nicely.

14. If we add the probabilities of exactly 4 hits, exactly 3 hits, exactly 2 hits, exactly 1 hit,
and no hits, we get _____. 1

Something is certain to happen!

In general,

$$(p+q)^n = \binom{n}{n}p^n + \binom{n}{n-1}p^{n-1}q^1 + \binom{n}{n-2}p^{n-2}q^2 + \dots \binom{n}{1}pq^{n-1} + \binom{n}{0}q^n .$$

By now, you realize why the probability distributions discussed in Section 9-5 are called binomial distributions.

Exercises. (Answers on page 331.)

1. Write the first 3 terms of $(x + y)^7$.
2. Find $(1.01)^5$ by using the fact that $1.01 = 1 + .01$.
3. Show how you could easily approximate $(1.02)^6$ to the nearest hundredth by using the binomial theorem.
4. Write as a sum, using the binomial theorem:
   (a) $(x + 2y)^3$
   (b) $(x - y)^4$
   (c) $(x + \frac{1}{x})^5$
   (d) $(x + \frac{1}{2})^3$
5. (a) Complete the following:

   $$(x + y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \underline{\qquad}$$

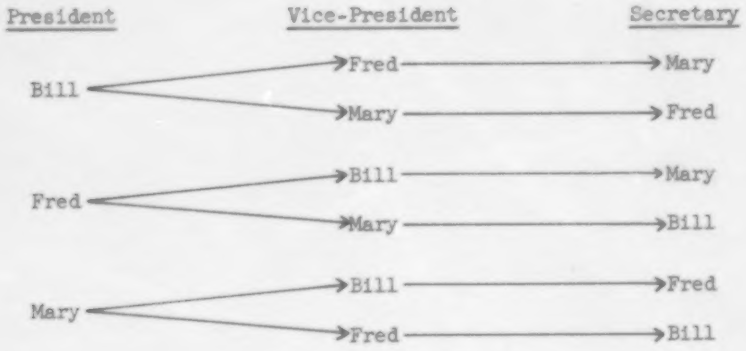
   (b) Let $x = 1$, $y = 1$. Use 5(a) to show that the sum of the terms in the 5th row of the Pascal triangle is 32.

   (c) Generalize this last result.

9-7. A Formula for $\binom{n}{k}$

Bill, Fred, and Mary are the only members of a club. They want the club to have three officers--president, vice-president, and secretary. In how many ways can officers be chosen?

Again, it is useful to make a tree diagram. If we know that Bill, Fred, and Mary are officers, then we can list the ways the offices can be distributed among them, as follows:


President
Vice-President
Secretary
Bill
Fred
Mary
Mary
Fred
Fred
Bill
Mary
Mary
Bill
Mary
Bill
Fred
Fred
Bill

3	1. As our diagram shows, any one of the ______ members may be president.
2	2. Once the president has been chosen, there are ______ possible choices for vice-president.
1	3. Once the president and vice-president have been chosen, there is only ______ way to choose a secretary.
$3 \cdot 2 \cdot 1$; 6	4. The officers can be chosen in $\underline{3 \cdot \square \cdot 1}$, or ______ ways.

Now let us suppose that a club of 8 members wishes to choose 3 to fill the same offices. Again we wish to find the number of ways officers can be chosen. Again we can think of how we would make a "tree".

8	5. There are ______ possible choices for president.
7	6. Once the president has been chosen, ______ choices remain for vice-president.
6	7. For each choice of president and vice-president, there are ______ choices for secretary.
$8 \cdot 7 \cdot 6$	8. The total number of ways of choosing 3 officers for this club is $8 \cdot \_\_\_ \cdot \_\_\_$, or 336.

Now we are going to do the problem in another way. The new way is more complicated, but will show us something interesting.

We think: In order to choose officers of our eight-member club, we might first simply choose 3 members to be officers. We could then decide which of these 3 is to be president, which vice-president, and which secretary.

$\binom{8}{3}$	9. The number of ways of choosing 3 of the 8 members to be officers is ______. (This is the number of 8-letter words in O (for officer) and N (for not-officer) with exactly 3 O's.)
$3 \cdot 2 \cdot 1$	10. Suppose we have selected 3 members. We can assign the posts of president, vice-president, and secretary to these particular 3 members in $3 \cdot \_\_\_ \cdot \_\_\_$ different ways. (Item 4) Hence, in all there are $(3 \cdot 2 \cdot 1) \cdot \binom{8}{3}$ ways of selecting officers.

Combining our results in Items 8 and 10, we see:

$$(3 \cdot 2 \cdot 1) \cdot \binom{8}{3} = 8 \cdot 7 \cdot 6.$$

Of course we can easily find $\binom{8}{3}$ from our table. But suppose we wanted to find $\binom{8}{3}$ and we didn't have a table. Suppose we didn't want to make a Pascal triangle in order to find $\binom{8}{3}$. Then we could use the result we have just obtained:

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}.$$

56

11. From this sentence, $\binom{8}{3}$ = ______, which is what we find from the table.

How could we find $\binom{10}{4}$ without a table? We might think: Suppose a club of 10 members wishes to elect 4 officers.

9

$10 \cdot 9 \cdot 8 \cdot 7$

12. There are 10 choices for president, and for each of them ______ choices for vice-president.

13. Thinking of how our tree of choices would look, we see that there are $\underline{10 \cdot 9 \cdot \square \cdot \square}$ ways in all of electing 4 officers.

Again we can do the problem in a second way.

$\binom{10}{4}$

$4 \cdot 3 \cdot 2 \cdot 1$

14. There are $\binom{\ }{\ }$ ways of selecting 4 of the 10 members to be officers.

15. There are $\underline{\square \cdot 3 \cdot 2 \cdot 1}$ ways of assigning the 4 officers to a group of 4 members.

16. There are, in all, $(4 \cdot 3 \cdot 2 \cdot 1) \cdot \binom{10}{4}$ ways of choosing 4 officers for a club of 10 members.

Combining our results of Items 13 and 16, we have:

$$(4 \cdot 3 \cdot 2 \cdot 1) \cdot \binom{10}{4} = 10 \cdot 9 \cdot 8 \cdot 7 .$$

Thus,

$$\binom{10}{4} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210.$$

Formulas like this are more easily written if we introduce a new symbol. We will write the product $3 \cdot 2 \cdot 1$ as $3!$, which is read "3 factorial". Thus, $3!$ is another name for 6.

$4! = 4 \cdot 3 \cdot 2 \cdot 1$, or 24.

$5\cdot4\cdot3\cdot2\cdot1$; 120	17. $5! = \underline{\square\cdot\square\cdot\square\cdot\square\cdot\square}$, or ______.
factorial	5! is read "5 ______".
5!	18. Notice that $5\cdot4! =$ ______!
$6\cdot5\cdot4$	19. $\frac{6!}{3!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1} = \underline{6\cdot\square\cdot\square}$.
$8\cdot7\cdot6$	20. $\frac{8!}{5!} = \underline{8\cdot\square\cdot\square}$.

We saw after Item 10 that $\binom{8}{3} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1}$.

Using Item 20, we now see that $\binom{8}{3} = \frac{8!}{3!\,5!}$.

Similarly, after Item 16 we saw that

$$\binom{10}{4} = \frac{10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1} = \frac{10!}{6!\,4!}.$$

The reasoning we used in connection with $\binom{8}{3}$ and $\binom{10}{4}$ can be applied to any entry in the Pascal triangle. Here are other examples.

$\frac{6!}{2!\,4!}$	21. $\binom{6}{4} = \frac{6!}{\underline{\square!\,\square!}} = \frac{6\cdot5}{2}$
$\frac{8!}{4!\,4!}$	22. $\binom{8}{4} = \frac{8!}{\underline{\square!\,\square!}} = \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2\cdot1}$
$\frac{7!}{1!\,6!}$	23. $\binom{7}{1} = \frac{\;!}{\underline{\square!\,\square!}} = \frac{7}{1}$

You should verify that your results in Items 21, 22, 23 do indeed give you the correct entries in the table of Pascal's triangle.

1; n	24. If n is a natural number, n! is a product of n successive natural numbers, where the smallest factor is ______ and the largest factor is ______.
6!	25. $6\cdot5! =$ ______! . Similarly, $2\cdot1! = 2!$ and
3!	$3\cdot2! =$ ______! .

	26. **Indeed, if n is any natural number,**
$(n+1)!$	$(n+1) \cdot n! =$ ______ !
1	27. 1! , of course, is ______.

We would like to have $1 \cdot 0! = 1!$, so that the formula in Item 26 would hold for $n = 0$. Hence, it is customary to define 0! as 1.

This may seem strange to you at first. However, as you work more with factorials you will find that the definition is helpful.

We can sum up our results: If n, r are numbers for which $\binom{n}{r}$ is defined:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad .$$

$\frac{9!}{3!\,6!}$	28. $\binom{9}{3} =$ ______. (Use factorials.)
$\frac{11!}{4!\,7!}$	29. $\binom{11}{4} =$ ______. (Use factorials.)
$\frac{6!}{6!\,0!}$; 1	30. $\binom{6}{6} = \frac{\square!}{\square!\,\square!} =$ ______.
$\frac{6!}{0!\,6!}$	31. $\binom{6}{0} =$ ______. (Use factorials.)

Items 30 and 31 are consistent with what we already know. The definition, $0! = 1$, makes our formula applicable to all the entries in Pascal's triangle.

45	32. $\binom{10}{2} = \frac{10!}{2!\,8!} =$ ______.
45	33. $\binom{10}{8} = \frac{10!}{8!\,2!} =$ ______.
$\frac{7!}{4!\,3!} = 35$	34. $\binom{7}{4} = \frac{7!}{\square!\,\square!} =$ ______.
$\frac{7!}{3!\,4!} = 35$	35. $\binom{7}{3} = \frac{\square!}{\square!\,\square!} =$ ______.

	Can you make the generalization for yourself?
$\frac{n!}{k!(n-k)!}$	36. $\binom{n}{k} =$ ______. (Use factorials.)
$\frac{n!}{(n-k)!\,k!}$	37. $\binom{n}{n-k} =$ ______. (Use factorials.)
	Hence, for any n, k $(0 \leq k \leq n)$ $\binom{n}{k} = \binom{n}{n-k}$.

Our last result is consistent with what we have observed about the symmetry of the entries in any given row of Pascal's triangle.

	38. In particular,
$\binom{n}{0} = 1$	$\binom{n}{n} = \binom{n}{\square} =$ ______.
$\binom{n}{1}$	39. $\binom{n}{n-1} = \binom{\ }{\ } = n$.
1000	40. $\binom{1000}{999} =$ ______.
1	41. $\binom{2783}{2783} =$ ______.
$190 \left(= \frac{20 \cdot 19}{2 \cdot 1}\right)$	42. $\binom{20}{18} =$ ______.

	Leave the following in factorial form.
$\frac{472!}{129!\;343!}$	43. $\binom{472}{129} =$ ______.
$\frac{1000!}{900!\;100!}$	44. $\binom{1000}{900} =$ ______.
$\frac{52!}{13!\;39!}$	45. $\binom{52}{13} =$ ______. (This is the total number of possible bridge hands.)
$\frac{52!}{5!\;47!}$	46. $\binom{52}{5} =$ ______. (This is the total number of 5-card hands.)

Items 43 to 46 emphasize that the actual computation of $\binom{n}{k}$ may be quite difficult for large n. You might compute the value of $\binom{52}{5}$. You will then realize the difficulty of "multiplying out" $\binom{52}{13}$, $\binom{1000}{900}$, or $\binom{472}{129}$. Fortunately, computational short-cuts and accurate methods of approximation exist. We shall not discuss these methods in this course.

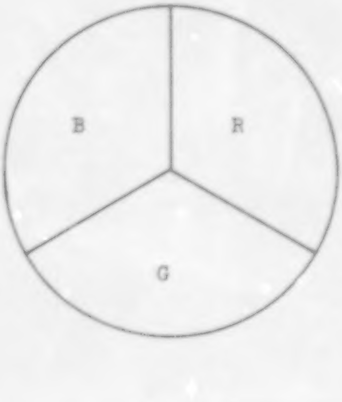
$\binom{7}{3}$ 35	47. In a certain class there are 7 boys. The teacher wishes to choose 3 of them to help in carrying some books. The teacher may select the boys in $\binom{\ }{\ }$, or ______ ways.
7 · 6 · 5; 210	48. Suppose the teacher wishes to choose boys to go on 3 different errands. One is to go to the library, one to the cafeteria, and one to the gym. This time the teacher may make the choice in $\underline{7 \cdot \square \cdot \square}$ = ______ ways.

The distinction between Items 47 and 48 should be noticed. In Item 47, the teacher makes an unordered selection. $\binom{7}{3}$ is often written as $C_{7,3}$ or as $_7C_3$ and is called the "number of combinations of 7 things, 3 at a time". In Item 48, the teacher makes an ordered selection. ("John, please go to the library. George, please go to the cafeteria. Max, please go to the gym.") 7 · 6 · 5 is often written as $P_{7,3}$ or as $_7P_3$ and is called the "number of permutations of 7 things, 3 at a time".

9-8. The Multicolored Spinner

Suppose we have a spinner divided equally into blue (B), green (G), and red (R) regions. Let us suppose that when we spin we are equally likely to get B, R, G. (If the spinner stops on a line, we don't count the spin.)

Suppose we spin 5 times, getting red, green, red, green, blue in this order.



RGRGB

1. The outcome for the 5 spins might be recorded:

   R ____ ____ ____ ____ .

   If we wish to list all possible outcomes for 5 spins, we can list all 5-letter "words" in the letters R, B, G.

3^5; 243

2. The number of possible outcomes is $3^{\square}$, or ______. (Think about how you could use a tree to make a list of possible outcomes.)

$\frac{1}{3^5}$, or $\frac{1}{243}$

3. Since all outcomes are equally likely, the probability of R G R G B is ______.

2

4. Suppose we wish to find the probability of getting exactly 2 reds and 2 greens in 5 spins. Then we must find the 5-letter words in which R and G each appear ______ times. (You found one example in Item 1.)

If we think about what we already know, it will save us trouble in counting.

3

5. Suppose we wanted only to find all the 5-letter words with 2 R's and 3 not-R's.

   We could do this easily. We would think of 5-letter words in R and N with R occurring 2 times and N ______ times.

$\binom{5}{2}$ or $\binom{5}{3}$; 10

6. There are $\binom{\ }{\ }$, or ______, such words.

   One such word is R N R N N.

B

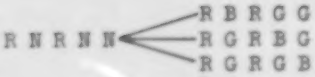
$\binom{3}{2}$ or $\binom{3}{1}$; 3

7. We can change this into a word with 2 R's, 2 G's, and 1 B by replacing 2 N's by G's and one N by ______.

   In how many ways can we choose which pair of N's to replace by G's ? $\binom{\ }{\ }$, or ______.

   Hence, to each word in R, N, with 2 R's, there are 3 words in R, G, B with 2 R's, 2 G's, and a B.

Example:



10 · 3; 30

8. Since there are 10 words of 5 letters in R, N with exactly 2 R's (Item 6), we conclude: There are ______ · ______, or ______, words of 5 letters with 2 R's, 2 G's, and 1 B.

The probability of each particular outcome with 2 R's, 2 G's, and 1 B is $\frac{1}{243}$. (See Item 3.)

$\frac{30}{243}$, or $\frac{10}{81}$

9. Hence, the probability of getting 2 R's, 2 G's, and 1 B is ______.

Exercises (Answers on page 331.)

1. Find the probability, for 5 spins of the spinner, of 2 R's and no G's; of 2 R's and 1 G; of 2 R's and 3 G's.
2. Use the results of (1) to find the probability that exactly 2 R's occur in 5 spins.
3. Show another method of finding the probability of exactly 2 R's in 5 spins.
4. Use a 3-colored spinner, as described in the problem, to see what results you actually get for 5 spins.

If you did Section 9-6, do Exercises 5, 6, and 7 below.

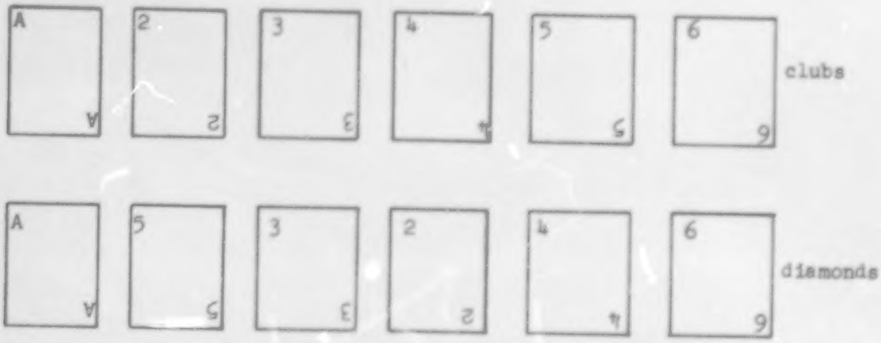
5. Expand: (a) $(x + y + z)^2$
   (b) $(x + y + z)^3$ Hint: $(x + y + z)^3 = [(x + y) + z]^3$
6. In the expansion of $(x + y + z)^4$, write the coefficient of x^3y, x^2y^2, and of xyz^2.
7. A 3-colored spinner (red, blue, green), each color having the same probability, is spun 4 times. What is the probability of getting 3 reds and 1 blue? Of 2 reds and 2 blues? Of 1 red, 1 blue, and 2 greens?

CHAPTER 10

MATHEMATICAL EXPECTATION

10.1 Experiment (Discussion is on page 307.)

For this experiment you may use some cards from a regular deck. Take the ace, 2, 3, 4, 5, 6 of clubs and of diamonds. Place the clubs on the table as shown. Now shuffle the diamonds well. One by one, place them on the table in the order they appear. Look below to see the way our cards looked when we did this.



Prepare a score sheet and score 1 for each pair that matches. Your score sheet should look like this:

Match / Trial	A	2	3	4	5	6	Total Number of Matches
1	1		1			1	3
2							
3							
4							
5							
6							
7							
8							
9							
10							

Our first trial--as shown--has been scored. You may use it. Then complete the table, performing the experiment 9 times. You must be sure to shuffle well between trials. It is difficult to shuffle 6 cards thoroughly. You will find it easier to shuffle if you add a few more cards. Simply ignore them when lining up the diamonds in the order they occur.

How many points did you score in 10 games? Approximately how many would you guess you would score in 100 games?

In this chapter we will learn more about techniques that can be used to help in analyzing problems of this kind. Save your record. We will refer to it in a later section.

10-2. The Mean

In this section we will review some ideas that you have used in situations having nothing to do with probabilities.

Example 1:

A student earned the following grades in a course:

$$83, 75, 92, 83, 83, 75, 76.$$

What mark can he expect for the course?

His mark will probably be based on his average. Let's compute the average grade:

$$83 + 75 + 92 + 83 + 83 + 75 + 76 = 567.$$

The average is $\frac{1}{7}(567) = 81$. Therefore, the student can probably expect a letter grade that corresponds to 81--probably a low B.

Instead of adding all scores, let's tabulate the grades differently.

He earned a 92 and a 76 just once; twice he got a 75 and three times an 83. So we can write:

Score	Number of Times Score Occurs	Computation of Average
92	1	$92 \cdot 1 = 92$
83	3	$83 \cdot 3 = 249$
76	1	$76 \cdot 1 = 76$
75	2	$75 \cdot 2 = \underline{150}$
		567

Again we obtain $\frac{1}{7}(567) = 81$ as the average grade. Looking at the grades, we could have seen that there were three grades between 70 and 80, and that four grades were above 80. So we could have guessed where the average would fall. It is always a good idea to estimate first and then compare the computed result with that guess.

Example 2:

On a certain test, students could get at most 80 points. What was the average test score if the following tabulation shows the results for 20 students?

Score	Number of Students	Computation
77	4	$4 \cdot 77 = 308$
74	7	$7 \cdot 74 = 518$
72	2	$2 \cdot 72 = 144$
70	5	$5 \cdot 70 = 350$
67	2	$2 \cdot 67 = 134$

1454 1. The sum of all the scores is ______.

72.7 2. The average score is $1454(\frac{1}{20})$, or ______.

The average is also called the arithmetic mean, or sometimes simply the mean. We will use the symbol m for the average, or mean. Thus, we write in this example:

$$m = 72.7 .$$

Let us look once again at the example we have just done. In order to find the average, or mean, we computed:

$$\left[4(77) + 7(74) + 2(72) + 5(70) + 2(67)\right]\left(\frac{1}{20}\right) .$$

We might have written this as:

$$77(\frac{4}{20}) + 74(\frac{7}{20}) + 72(\frac{2}{20}) + 70(\frac{5}{20}) + 67(\frac{2}{20}).$$

We see that the average grade can be found by multiplying each grade by the fraction of students getting this grade. We "weight" each grade by multiplying it by the appropriate fraction.

$\frac{4}{20}$	3. The fraction of students getting 77 is $\frac{\square}{20}$,
$\frac{1}{5}$	or ______.
$\frac{7}{20}$	4. The fraction of students getting 74 is ______.
1	5. Note that $\frac{4}{20} + \frac{7}{20} + \frac{2}{20} + \frac{5}{20} + \frac{2}{20} =$ ______.
no	6. The average score is 72.7. Did any student actually get the average score on his test paper? ______
11	7. How many students had scores above the average? ______
9	8. How many had scores below the average? ______

9. On a certain test 9 students got 100 and 6 students got 80. Is the average score for the class 90 ?

[A] Yes [B] No

Although the mean, or average, of the two numbers 100 and 80 is 90, this is not the average grade. You must "weight" each score by the fraction of students obtaining it:

$$m = 100(\frac{3}{5}) + 80(\frac{2}{5}) = 92.$$

Thus, [B] is the correct result.

Exercise. (Answers on page 332.)

1. Look back at your record for the experiment in Section 10-1. On the 10 trials, what is the average number of matches?

2. Look back at your record of 100 throws of a die.

   (a) For these throws, what is the average of the numbers thrown?

   (b) What was your average for the first 25 throws? For the last 25 throws?

3. A certain machine is designed to cut metal strips 3 inches long. In order to study the machine's accuracy, one day's output is examined carefully. It is found that $\frac{1}{10}$ of the strips are 2.8 inches long, $\frac{3}{10}$ are 2.9 inches long, $\frac{2}{5}$ are 3.0 inches long, $\frac{1}{5}$ are 3.1 inches long.

   (a) What is the average length of a strip?

   (b) If 1000 strips are produced during the day, how many are 2.8 inches long?

   (c) What is the sum of $\frac{1}{10}$, $\frac{3}{10}$, $\frac{2}{5}$, $\frac{1}{5}$?

4. A certain golfer says, "I have played golf 15 times this year. On the average, I have lost 2.6 balls per game." How many balls has the player lost in all?

10-3. Mathematical Expectation: Definition

In Exercise 2, Section 10-2, we computed the average of the numbers thrown in 100 throws of a die. Suppose you are going to throw a die a great many times. What is your guess about the average of the numbers thrown? Examine your results in the exercise, and also think about how you might justify your guess. Then read on.

$\frac{1}{6}$

1. You might reason: Suppose a die is thrown many times. Since all 6 faces are equally likely, we expect each number to occur approximately ______ of the time.

$5(\frac{1}{6}) + 6(\frac{1}{6})$

2. Hence, in order to guess the average of the numbers thrown, it is reasonable to compute

$$1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + \_\_\_\_ + \_\_\_\_.$$

$\frac{21}{6}$, or 3.5

3. Our result, then, is $\frac{\square}{6}$, or ______.

Notice that in the sum in Item 2, each of the numbers 1, 2, 3, 4, 5, 6 has the same "weight", $\frac{1}{6}$. This is reasonable, since all these numbers are equally likely.

Let us consider a second example. Suppose we toss 3 coins many times, recording for each toss the number of heads that occur. (We have already considered situations of this type in earlier chapters.)

$\frac{1}{8}$

4. We know, for example, that the probability of getting no heads on a toss is ______. (If you had trouble, make a tree or list the possible outcomes--HHH, HHT, etc.)

5. Complete the following table.

Number heads	0	1	2	3
Probability	$\frac{1}{8}$			

Compare your results with the table below.

Number heads	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\frac{3}{8}$

6. If you toss three coins many times, you guess that you would get exactly two heads about ______ of the time.

$2(\frac{3}{8}) + 3(\frac{1}{8})$

$\frac{3}{2}$, or 1.5

7. You guess that the average number of heads would be

$$0(\tfrac{1}{8}) + 1(\tfrac{3}{8}) + \_\_\_\_\_\_ + \_\_\_\_\_\_,$$

or ______.

Note that your result in Item 7 looks like sums we used in computing averages in Section 10-2.

In this section, we have considered two examples. They have many elements in common.

In each, we are thinking of an experiment--throwing a die; tossing three coins. In each, every outcome of the experiment yields a number, and it is these numbers we are considering.

6

8. Each throw of a die yields exactly one of the numbers 1, 2, 3, 4, 5, or ______.

3

9. Each toss of three coins yields, as the number of heads shown, one of the numbers 0, 1, 2, or ______.

Moreover, in each example, we know the probability of getting each of the numbers involved. For the die we have

Number shown	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

For the three coins, we have:

Number heads	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(If you have completed Chapter 9, these tables will be familiar to you.)

In each case, our guess as to the average was derived by multiplying each number by its probability and then adding. The result is called the mathematical expectation.

$\frac{7}{2}$, or 3.5	10. If we throw a die, the mathematical expectation of the number shown is ______. (Item 3.)
$\frac{3}{2}$, or 1.5	11. If we toss three coins, the mathematical expectation of the number of heads is ______. (Item 7.)
expectation	12. If each outcome of an experiment gives a number, then the weighted mean of these numbers, weighted by the probabilities, is the mathematical ______.

Sometimes the mathematical expectation is called the expected value. This phrase is a little easier to say, but we must be careful to understand it.

	We do not always "expect" to obtain the expected value.
$\frac{3}{2}$	13. For example, the expected value of the number of heads if three coins are tossed is ______. (Item 11.)
cannot	14. We ______ (can,cannot) obtain $\frac{3}{2}$ heads in a toss of 3 coins.
is	15. However, if we perform the experiment many times, the average of the numbers we get ______ (is,is not) likely to be close to the mathematical expectation.

The symbol M is often used for the mathematical expectation.

Note that we used m for the average of numbers resulting from performing an experiment several times. M, however, is used when we replace experimental data with probability. Whether M or m, we should be reminded of mean--weighted average.

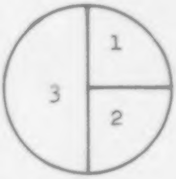
Exercises. (Answers on page 333.)

1. At a party, prizes are drawn from a pack. In the pack, there are 6 prizes that cost \$1.00. There are 9 that cost \$.60. There are 5 that cost \$.20. A guest draws a prize.

   (a) Again, we have an experiment--drawing a prize--and a set of numbers that can result. Make a table showing the numbers and their probabilities.

   (b) Find the expected value of the draw.

2. A game is played with the spinner shown. A player spins and moves the number of spaces indicated.



   (a) Make a table showing the possible numbers of spaces and their probabilities.

   (b) What is the expected value of the number of spaces moved?

3. (a) One coin is tossed. What is the expected value of the number of heads?

   (b) Two coins are tossed. What is the expected value of the number of heads?

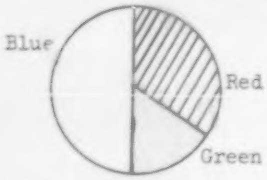
   (c) Referring to (a), (b), and using also Item 7, guess the expected value of the number of heads if 4 coins are tossed.

4. At a certain intersection it is known that the number of accidents during a given hour on a certain day have the following probabilities:

No. of accidents	0	1	2	3
Probability	.94	.03	.02	.01

Find M, the expected number of accidents during that hour on one such day. (Just compute, do not experiment!) How could the answer be best interpreted?

10-4. Mathematical Expectation: Interpretation



Here is a spinner, for which $P(\text{red}) = \frac{1}{3}$, $P(\text{green}) = \frac{1}{6}$, $P(\text{blue}) = \frac{1}{2}$.

Suppose you are playing a game, in which you get points as follows:

Red - 5 points
Green - 2 points
Blue - 1 point.

$2(\frac{1}{6})$
$2\frac{1}{2}$

1. The expected value of the score is $5(\frac{1}{3})$ + _____ + $1(\frac{1}{2})$, or _____.

No

2. Is it possible to obtain a score of $2\frac{1}{2}$ on any particular spin? _____.

average

3. If you played many times, your _____ score would be approximately $2\frac{1}{2}$.

One way to interpret expected value is to think of a game of chance. Suppose you were planning to play this game many times. Suppose that it cost you $2.00 each time you played, and that you got $1.00 per point you won. Then you might expect to gain, on the average, $.50 a game. But if you had to pay $3.00 per game, you would expect to lose $.50 a game on the average.

Here is another example.

On a second game, played with the same spinner, you get points as follows:

Red - 4 points
Green - 3 points
Blue - 1 point.

Let us suppose you are offered a choice. You may play either game, and both games cost $2.00. Which game do you prefer?

$2\frac{1}{3}$ 4. Your expected value for number of points won in this game is ______.

less 5. You gain a little ______ (more,less), in the long run, on this game than on the first.

less; more 6. This is a reasonable conclusion. Comparing the rules, we see that on the second game we get 1 point ______ (more,less) for each red spin and 1 point ______ (more,less) for each green spin.

more 7. But red is ______ (more,less) likely than green.

Hence, our expectation is less in the second game.

Explaining ideas about probability in terms of games may seem somewhat artificial. You may not want to play games for money. In fact, common sense by now should tell you that the roulette wheels at Monte Carlo, for example, are arranged so that the amount you pay is greater than the expected value of your winnings.

However, situations arise in which decisions must be made, even though the future is uncertain. Toy manufacturers, for example, cannot predict with certainty which items will be most popular. It is to their advantage to produce amounts of various items which make the expected value of their profits as large as possible.

Here are two games, each played by throwing a die once.

A: You get 21 points if 6 is thrown, otherwise you get nothing.

B: You win the number of points thrown.

$\frac{21}{6}$, or $3\frac{1}{2}$ 8. The mathematical expectation for game A is ______.

$3\frac{1}{2}$ 9. The mathematical expectation for game B is ______. (You have derived this result earlier.)

Suppose you could play one of these games, and that in each you would win a dollar for each point. Which game would you choose? Look back at Items 8 and 9 and consider carefully.

The best answer that can be given is--it doesn't matter. The two games have the same expected value. If half your class made one choice and half the other, there is no reason to think one group's gains would exceed the other--in the long run. After, say, 1000 tosses, each group would expect to have accumulated $3,500.

Some people accept this argument readily but still raise the question: If there is to be only one toss, is one game better than the other?

You may feel that you would prefer to be sure of winning something. You would then choose the first game, though in it you can't win more than $6.00, thus surrendering the possibility of receiving $21.00. On the other hand, you may feel that it would be better to "take a chance" on winning $21.00, even though the probability of receiving nothing is $\frac{5}{6}$.

In either case, your decision is based on personal feelings, not reflected in the mathematical expectation. The expected value of a single throw is $3.50 for both games A and B.

There are many important practical situations in which factors such as personal feelings affect decisions, regardless of the mathematical expectations. Considerations of this sort have led economists and others to introduce the idea of "utility", a concept which is needed when dollar values do not accurately reflect an individual's feelings.

Exercises. (Answers on page 333.)

1. A certain king is always at war with his neighbors. In fact, once a year his neighbors attack either town A or town B. The king has to decide whether to station all his army at town A or at town B. If the king has all his army stationed at town A, he finds that an attack does little damage--it costs only one bag of gold to repair. If the army is at B, however, when B is attacked, there is no damage at all. But if A is attacked when the army is not there, then repairing A costs 4 bags of gold, while if B is attacked when the army is not there, then repairs to B cost only 2 bags of gold. In order to decide whether to station his army at A or B, the king sends spies to find out which town will be attacked. What should he decide if his spies report:

(a) attacks on A and on B are equally likely?

(b) an attack on B is 4 times as likely as an attack on A ?

(c) an attack on A is $\frac{2}{3}$ as likely as an attack on B ?

Suggestion: In this kind of problem it helps to use a table to summarize the facts. Here is a table:

Enemy attacks

		A	B
Army is at:	A	1	2
	B	4	0

Cost to the king
in bags of gold

2. Two dice are thrown.

(a) Make a probability table for all possible sums.

(b) What is the expected value of the sum when two dice are thrown?

3. Again two dice are thrown, one red and one green.

(a) Guess the expected value of the difference:

number on red - number on green.

(Note that the difference may be negative.)

(b) Verify your guess by computing the expected value.

4. Here is another game played with a red and a green die. The dice are thrown. One player gets the product of the numbers thrown. The other player gets the square of the number on the red die.

(a) Guess: Which player has the advantage- that is, which expects to win the most points?

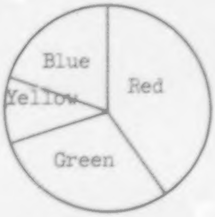
(b) Compute the mathematical expectation for each player. Was your guess right?

(Remember your results for this exercise. We will refer to them later.)

10-5. The Mathematical Expectation of a Sum

A certain king had two sons, Andrew, who was the older, and Bruce. He wished to share his treasure with them. He said: "Here is a spinner. On it, P(red) = .4, P(green) = .3, P(yellow) = .1, and P(blue) = .2. I will spin it 1000 times. On each spin I will award gold pieces, as follows:

Spin	Red	Green	Yellow	Blue
To Andrew	20	20	50	50
To Bruce	30	0	40	10
Probability	.4	.3	.1	.2



In this way, each of you will have a fortune." How big a fortune?

Bruce at once computed the expected value of his gain on a single spin.

He used only this part of the king's table:

Gain	30	0	40	10
Probability	.4	.3	.1	.2

10(.2) 1. Bruce's expected value = 30(.4) + 0(.3) + 40(.1) + ____( ),
18 or ______, gold pieces.

2. He would expect to average 18 gold pieces per spin.
18,000 In 1000 spins, he would expect about ______ gold pieces.

3. Andrew computed his expected value as follows:

Expected value =

50(.2) 20(.4) + 20(.3) + 50(.1) + ____( ),
29 or ______.

29,000 4. In 1000 spins, Andrew would expect about ______ gold pieces.

20(.7)

5. Notice that Andrew could have simplified his work by noting that $20(.4) + 20(.3) = \underline{20(\quad)}$.

50(.3)

6. Likewise, $50(.1) + 50(.2) = \underline{50(\quad)}$.

distributive

7. Items 5 and 6 illustrate the ______ property.

How much, on the average, does the king give away on each spin? We might reason, simply, he averages 29 gold pieces per spin to Andrew, and 18 per spin to Bruce.

47 (29 + 18)

8. It seems reasonable to guess:

Expected value of gift, per spin = ______.

In other words, let:

M_G = expected value of king's gift

M_A = expected value of Andrew's gain

M_B = expected value of Bruce's gain.

On each spin, we have a value of G, A, B, and for these values

$$G = A + B.$$

It seems reasonable, therefore, that:

$$M_G = M_A + M_B.$$

In order to understand this relation better, we shall compute M_G, the expected value of the king's gift, in a special way. We will use this table:

Gift: $G = A + B$	$20 + 30$, or 50	$20 + 0$, or 20	$50 + 40$, or 90	$50 + 10$, or 60
Probability	.4	.3	.1	.2

$$\text{Expected value} = M_G = (20 + 30)(.4) + (20 + 0)(.3) + (50 + 40)(.1) + (50 + 10)(.2).$$

30(.4)	9. But we know that (20 + 30) .4 = 20(.4) + ___( ).
distributive	10. Again, we have used the ______ property.

Hence, we see:

$$M_G = 20(.4) + 30(.4) + 20(.3) + 0(.3) + 50(.1) + 40(.1) + 50(.2) + 10(.2) \ .$$

Using the associative and commutative properties, we rearrange terms.

$$\begin{aligned} M_G &= [20(.4) + 20(.3) + 50(.1) + 50(.2)] \\ &\quad + [30(.4) + 0(.3) + 40(.1) + 10(.2)] \\ &= M_A + M_B. \end{aligned}$$

This same reasoning can be used in any problem where we wish to find the expected value of a sum.

Hence, we always have: The mathematical expectation of the sum can be found by adding the separate mathematical expectations.

We have already seen some examples which illustrate this result.

3.5	11. For a throw of a single die, we found: The expected value of the number thrown is ______.
7	12. We also found (Section 10-4, Exercise 2) that for a throw of 2 dice the expected value of the sum is ______.
10.5	13. It appears reasonable to suppose that for 3 dice the expected value of the sum is ______.
	This is true: We add the expected value for each die to find the expected value of the sum. The reasoning is exactly like that in the previous example.

	Look back at your results in Section 10-3, Exercise 3.
$\frac{1}{2}$	14. If you toss a single penny, the expected value of the number of heads is ______.
$\frac{1}{2} + \frac{1}{2}$; 1	15. If you toss a penny and a dime, the expected value of the number of heads is ___+___, or ______.

We can apply our knowledge about the expected value of the sum to Experiment 10-1, which had to do with matching 6 cards.

This experiment illustrates the famous problem of "matching" (rencontre), which goes back to the Montmort (1678-1719). It was generalized by Laplace (1749-1827) and others. The problem takes on various forms. Here is one example: 75 personal letters are written and the envelopes addressed. The envelopes drop on the floor, get all mixed up, and, without sorting them, the "inefficient" secretary places each letter in an envelope. What is the probability that any given letter and envelope are matched properly?

Look back at your records for Experiment 10-1. We found the average number of matches, but we did not find the probabilities involved, for a very good reason. To compute directly the probabilities of no matches, one match, etc., we would need to think about all the ways of arranging 6 cards. From Chapter 9, we know that there are 720 ways. In order to find the probabilities for our matching problems with 6 cards, we need shortcuts, and these shortcuts are beyond the scope of this chapter.

However, it often occurs in mathematics that a simple situation throws light on a more complicated one. So, let us look at what happens when the number of cards is smaller.

Exercises: (Answers on page 336.)

1. With 1 card:
   (a) Find P(no matches), P(one match),
   (b) What is the expected value of the number of matches?

2. With 2 cards, find:
   (a) P(no matches), P(one match), P(2 matches);
   (b) Expected value of number of matches.

3. With 3 cards, find:
   (a) the probabilities of 0, 1, 2, 3 matches;
   (b) expected value of number of matches.

4. Look at your results for the expected values in Exercises 1 to 3 and in Experiment 10-1. Can you make a guess about something that is true for any number of cards?

Let us see whether, for 6 cards, we can find the mathematical expectation for the number of matches without actually computing the probability of no matches, of one match, etc. Look back at the experiment, and look at your record for it.

$\frac{1}{6}$

16. Any of the 6 cards is as likely as any other to be put down first. Thus, it is reasonable that there should be a match on the first card about ______ (fraction) of the time.

$\frac{1}{6}$

17. Thus, in the first column of your record you would see a 1 recorded in about ______ of the spaces.

The expected value of the entry in the first column is therefore, $1 \cdot \frac{1}{6}$, or $\frac{1}{6}$.

$\frac{1}{6}$

18. By similar reasoning, the expected value of the entry in the second column is ______.

The number of matches for any single trial is found by adding the entries in the separate columns for that trial. Hence, the expected value of the number of matches is found by adding the expected values of the column entries.

1

19. Expected value of number of matches = ______.

1

20. This is true because:

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 6(\tfrac{1}{6}) = \underline{\qquad}.$$

21. Would the same reasoning apply if we used any other number of cards?

[A] Yes [B] No

The argument holds for any number of cards. For 10 cards, for example, you would suppose that 1 would appear in each of the 10 columns about $\frac{1}{10}$ of the time. [A] is correct.

Here is another example:

22. You and a friend are going to toss coins. Whenever the results match, you score 1 point. When they do not, you score 0 points. If you are going to toss 10 times, what is the expected value of your score?

5 ______.

Do you see how we have applied, once again, the same reasoning?

23. Think about any single trial--that is, pair of tosses. Whatever your friend's result, the probability that

$\frac{1}{2}$ yours will match it is ______.

24. The expected value of your score on a single trial

$\frac{1}{2}$ is $1(\frac{1}{2}) + 0(\frac{1}{2}) =$ ______.

Your score for 10 trials is found by adding your score for the separate trials.

Hence, the expected value of your score is found by adding the ten separate expected values.

$10(\frac{1}{2})$, or 5 — 25. The expected value of your score is ______.

We can think of many problems relating to matching.

Let us return to the inefficient secretary who was introduced at the beginning of this section.

26. Out of 75 letters, what is the probability that a particular letter is matched with the proper

$\frac{1}{75}$ envelope? ______.

Note: For a particular letter there are 75 possible envelopes--all equally likely.

We might ask: What is the probability that at least one letter is placed in the correct envelope?

In seeking to answer this question, you might note that from your results in Exercises 1, 2, 3, we know the probabilities if she had had, instead of 75 letters, only 1, or 2, or 3.

For one letter, P(at least one match) = 1

For two letters, P(at least one match) = $\frac{1}{2}$

For three letters, P(at least one match) = $\frac{2}{3}$.

You might then try 4 letters, 5 letters, etc., looking for a pattern as you do so. Unfortunately, the counting process becomes lengthy without some additional tools. Therefore, we will simply tell you the result. It turns out that as you increase the number of letters, the probabilities change. (For instance, with 4 the probability of at least one match is $\frac{5}{8}$.) For 75 letters the probability that at least one letter and envelope will match is close to $\frac{2}{3}$.

Surprisingly, the result would be approximately the same if there were 100, or 1000, or 10,000 letters--with, of course, the same number of envelopes.

We can restate this problem in many ways. Suppose, for example, we have two identical sets of 100 cards. If each set of cards is well shuffled and two people each turn one card at a time, what is the probability that both will turn up the same card at least once?

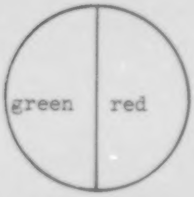
Again, the probability is approximately $\frac{2}{3}$.

10-6. Decision Plans

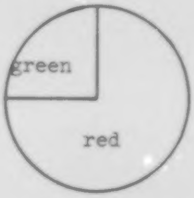
As we have observed, it is very often necessary in practical situations to make decisions even when we cannot be certain we are right. Knowing probability does not make us certain, but it gives us better ways of estimating the likelihood that we are right.

Consider these two spinners.

I.



II.



$P(\text{red}|\text{I}) = \frac{1}{2}$

$P(\text{green}|\text{I}) = \frac{1}{2}$

$P(\text{red}|\text{II}) = \frac{3}{4}$

$P(\text{green}|\text{II}) = \frac{1}{4}$

Suppose a friend tells us: I am going to toss an honest coin. If it lands heads, I will spin I. Otherwise, I will spin II. I am going to tell you whether I got red or green. You are to guess which spinner I used.

We know, of course, that $P(\text{I}) = \frac{1}{2}$ and $P(\text{II}) = \frac{1}{2}$.

Using methods developed in Chapters 7 and 8 we can show:

$P(\text{I}|\text{red}) = \frac{2}{5}$

$P(\text{II}|\text{red}) = \frac{3}{5}$

$P(\text{I}|\text{green}) = \frac{2}{3}$

$P(\text{II}|\text{green}) = \frac{1}{3}$

$P(\text{I} \cap \text{red}) = \frac{1}{4}$

$P(\text{I} \cap \text{green}) = \frac{1}{4}$

$P(\text{II} \cap \text{red}) = \frac{3}{8}$

$P(\text{II} \cap \text{green}) = \frac{1}{8}$

(If you do not understand how these probabilities were derived, pause here and do Exercise 10 at the end of this section.)

Suppose we make, in advance, a plan for deciding which spinner was used.

$<$	1. We note that $P(I\|red)$ ______ $P(II\|red)$. (<, =, >)
II	2. Hence, we plan: If our friend says "red", we will decide ______. (I,II)
$>$	3. But $P(I\|green)$ ______ $P(II\|green)$. (<, =, >)
I	4. Thus, we plan: If our friend says "green", we decide ______.
	5. But we recognize that we may decide wrong! Indeed, we decide wrong whenever either:
I green	our friend spins red on ______, or our friend spins ______ on II.
$II \cap green$	6. Hence, $P(\text{we are wrong}) = P(I \cap red) + P(\_\_\_\_)$ $= \frac{3}{8}$.

If we repeat this procedure many times, we will be wrong about $\frac{3}{8}$ of the time. However, our plan, based on the information in Items 1 and 3, is better than some of the other plans we might have chosen.

$\frac{1}{2}$	7. For example, if we always decide I, regardless of what our friend says, we will be wrong about ______ of the time.
$\frac{1}{2}$	8. Suppose we plan that we will decide by tossing a coin. If we get heads, we will decide on I; if tails, on II. We will be wrong about ______ the time. (If you had trouble, look back at Items 22 to 25, Section 10-5.)

Now let us suppose our friend tells us: I will toss a coin, as before. Again, I will select spinner I if I get heads, and II if I get tails. This time I will spin the spinner 100 times. I will tell you how many times I get red. Then, as before, you are to decide which spinner I was using.

9. When you analyze this situation, you note that for spinner I the expected value of the number of reds
50 in 100 spins is ______.

10. On the other hand, the expected value of the number of
75 reds for spinner II is ______.

11. We find these values, of course, by noting that for a single spin the expected value of the number of reds is

$\frac{1}{2}$ ______ for spinner I ;

$\frac{3}{4}$ ______ for spinner II.

Intuitively, we feel that for 100 spins of I the number of reds is usually about 50. In fact, it turns out that 50 is the most likely number of spins, 49 and 51 the second most likely numbers. Similarly, we expect about 75 reds if spinner II is spun 100 times.

Thus, our decision plan might be:

decide I if the number of reds is 62 or less;
decide II if it is more than 62.

The probability is very great--we feel--that when our friend reports his results one of two things occurs. Either he reports approximately 50 reds (and we decide I) or he reports approximately 75 reds (and we decide II). Usually one of these will happen, and usually our decision in either case will be correct.

Of course, if our friend reports 61 reds, we are not going to be very confident about our decision. However, we feel that it is very unlikely that he will report 61 reds.

Our intuitive feelings here are quite correct. Our probability of deciding correctly is much greater if the results of 100 spins are known than if we know only about 1 spin. It is greater still if the number of spins is further increased.

We observe that it is of interest to consider not only the expected value itself but the tendency of results to cluster around the expected value. Though a full discussion of this is beyond the scope of this book, the next section of this chapter deals with a technique that is useful in this connection.

If you have completed Chapter 9, you should do Exercises 1 through 9, which are based on results obtained in that chapter.

Exercises.

(Answers on page 337.)

1. In Chapter 9 we considered the method for finding the probability of exactly k successes in n independent trials, where the probability of success on each trial is p. Even if you have forgotten the formula for the probabilities, it is easy to see that the expected value of the number of successes is np. Explain. (Hint: What is the expected value of the number of successes for a single trial?)

2. Refer to the bar graph on page for $n = 4$, $p = \frac{1}{2}$.
   - (a) What is the expected value of the number of successes?
   - (b) What is the most probable number of successes?
   - (c) What is the probability that the number of successes differs from the expected value by at most 1 ?

Answer the same questions as in Exercise 2 for each of the following cases. The bar graphs are on the pages indicated.

3. $n = 4$, $p = \frac{1}{3}$. (page 200)

4. $n = 4$, $p = .8$. (page 200)

5. $n = 4$, $p = .2$. (page 200)

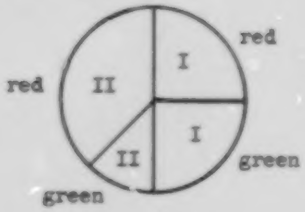
6. $n = 4$, $p = .6$. (page 200)

7. $n = 6$, $p = \frac{1}{2}$. (page 203)

8. $n = 8$, $p = \frac{1}{2}$. (page 203)

9. $n = 10$, $p = \frac{1}{2}$. (page 203)

10. Consider the following spinner.



$P(\text{red I}) = P(\text{green I}) = \frac{1}{4}$

$P(\text{red II}) = \frac{3}{8}$

$P(\text{green II}) = \frac{1}{8}$

(a) Find $P(\text{red}|\text{I})$, $P(\text{I}|\text{red})$, $P(\text{II}|\text{red})$.

(b) Compare the probabilities for this spinner with those for the two spinners at the beginning of this section.

10-7. Standard Deviation

Example. A teacher gives a test to three groups of students. Here are the results.

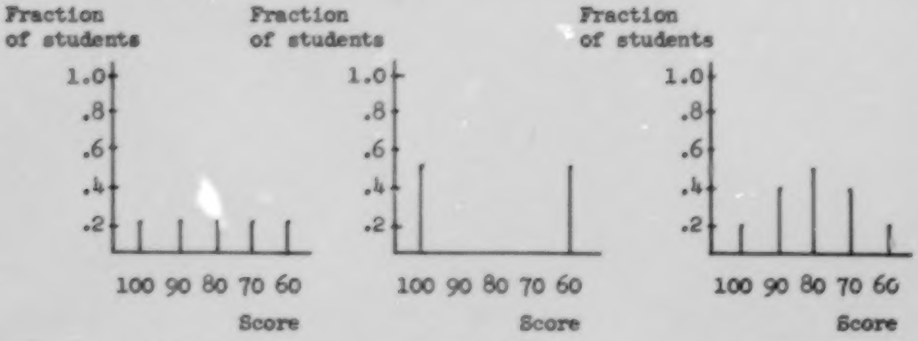
Score	100	90	80	70	60	50
Group I	2	2	2	2	2	0
Group II	5				5	0
Group III	1	2	4	2	1	0

80 1. The average grade for Group I is ______.

80 2. For Group II and Group III, the average is also ____.

Though all three groups have the same average, they differ in certain ways.

These differences can be visualized by drawing bar graphs.



Group I Group II Group III

In Group III, we observe that most of the grades are fairly near the mean, 80. In Group II, however, none are near 80. The grades in Group II are most spread out; those in Group III are the least so.

It is useful to have, for each graph, a number which measures the amount of spreading. Our next goal will be to define such a number.

Our first thought might be to find, for each score, the distance between the score and the mean. This distance can be thought of as the deviation of the score from the mean. We could then average these deviations, and we could use this average as a measure of the spread, or dispersion, of the scores.

It turns out, however, that it is more desirable to use instead a slightly more complicated measure of dispersion. This measure is found as follows:

1. For each number in the table (each score in our examples), find the difference between it and the mean m.
2. Square each difference.
3. Multiply each square by the fraction of the group associated with the number (the score).
4. Add the products obtained in 3.
5. Take the square root of the result. This square root is the standard deviation.

Some examples will make the procedure clear.

Finding the standard deviation for the scores in Group I.

Score	100	90	80	70	60
Number obtaining score (f)	2	2	2	2	2
Fraction $(\frac{f}{10})$	.2	.2	.2	.2	.2
(1) Score - m	20	10	0	-10	-20
(2) $(\text{Score} - m)^2$	400	100	0	100	400
(3) $(\text{Score} - m)^2 \frac{f}{10}$	400(.2)	100(.2)	0(.2)	100(.2)	400(.2)

Adding across line (3), we have:

(4) $400(.2) + 100(.2) + 0(.2) + 100(.2) + 400(.2) = 200$

(5) Now, $\sqrt{200} \approx 14.14$.

The standard deviation of the scores of Group I is approximately 14.14.

Examine the preceding work. Complete the items in this box to test your understanding.

3. Under the score 100, we see

20 score - m = ______, and

400 $(\text{score} - m)^2$ = ______.

4. Under the score 60, we see

-20 score - m = ______, and

400 $(\text{score} - m)^2$ = ______.

$(-20)^2$ 5. $(60 - 80)^2 = \underline{(-\quad)}^2$

400 = ______.

Note that all the entries in the bottom row are non-negative, since each is the square of a real number.

We have used f for the number of times the score occurs. The number of times of occurrence is often called the frequency. The table of test scores and frequencies is sometimes called a frequency distribution.

Let us find the standard deviation for the scores in Group II.

6. Make a table, as in the example. Compare your result carefully with the one below.

Score	100	90	80	70	60
f	5	0	0	0	5
$\frac{f}{10}$	.5	0	0	0	.5
score - m	20	10	0	-10	-20
$(\text{score} - m)^2$	400	100	0	100	400
$(\text{score} - m)^2 \frac{f}{10}$	200	0	0	0	200

7. Adding across the last line, we find the sum is ______.

400

8. Standard deviation = $\sqrt{\quad}$ = ______.

$\sqrt{400}$, 20

10.95

9. The standard deviation for Group III is ______.

If you had difficulty, compare your work carefully with the result below.

Score	100	90	80	70	60
f	1	2	4	2	1
$\frac{f}{10}$	.1	.2	.4	.2	.1
score - m	20	10	0	-10	-20
$(\text{score} - m)^2$	400	100	0	100	400
$(\text{score} - m)^2 \frac{f}{10}$	40	20	0	20	40

$$40 + 20 + 0 + 20 + 40 = 120.$$

Standard deviation = $\sqrt{120}$ = 10.95 .

We have found the standard deviations in three examples:

For Group I, standard deviation = 14.14
For Group II, standard deviation = 20.00
For Group III, standard deviation = 10.95.

Look back at the graphs we have drawn. It appears in these examples that the largest standard deviation goes with the graph with greatest spread.

Example

In the Stanford Coaching Camp, Stanford, California, the heights and weights of 32 boys were recorded in July, 1965. The boys were to enter grade 9 in September, 1965. In the distribution for heights, we will use the symbols h for the number of inches in the height and w for the number of pounds. (We listed the heights and weights in vertical columns simply for ease in reading.)

Height in Inches	Frequency
h	f
55	1
58	1
59	1
60	1
61	3
62	2
63	3
64	4
65	2
66	3
68	3
69	3
70	2
71	1
73	2

Weight in Pounds	Frequency
w	f
165	2
159	1
150	2
148	1
138	2
130	5
126	1
120	2
115	1
112	3
106	1
105	4
100	1
98	1
95	3
87	1
75	1

Exercises. (Answers on page 338.)

1. Find the mean (average) height and weight.
2. Find the standard deviation for the heights.

We have graphed the frequency distributions for the heights and weights. Examine the graphs carefully. We have indicated, on each, the mean. We have also indicated a band around the mean, extending each way from the mean a distance of 1 standard deviation.

The computation of the standard deviation--22.7--for the weights is tedious. We used a desk calculator to find it. You may wish to check our result.

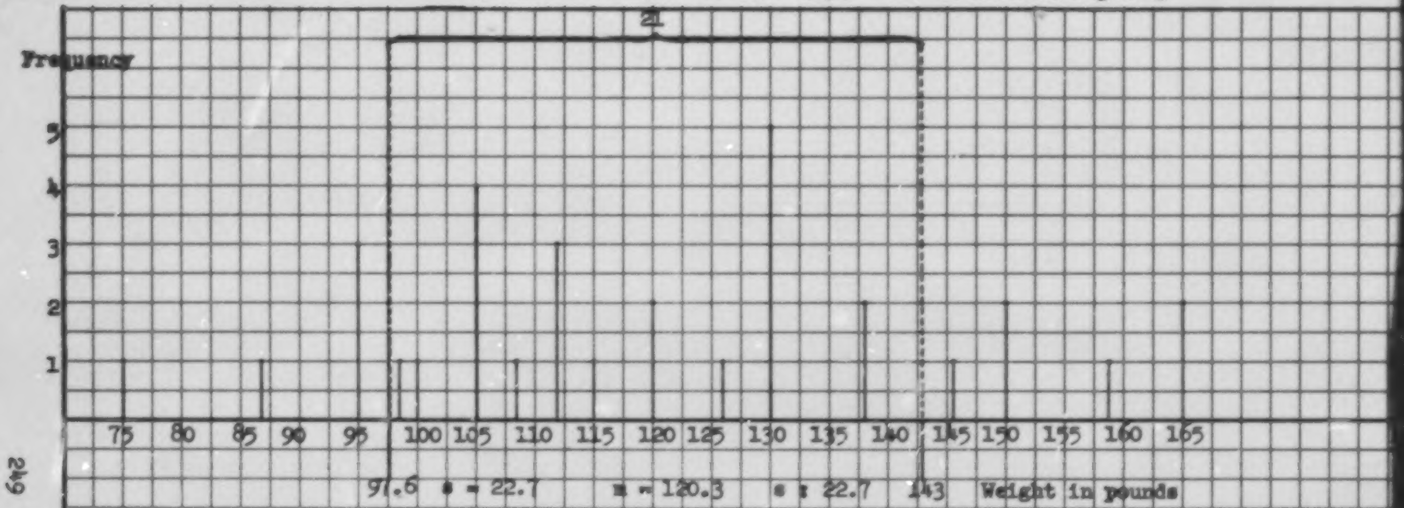
Notice that $\frac{23}{32}$ of the heights are within 1 standard deviation of the mean, while $\frac{21}{32}$ of the weights are within 1 standard deviation of the mean.

In many situations we find--as in these examples--that about $\frac{2}{3}$ of all values lie within 1 standard deviation of the mean.

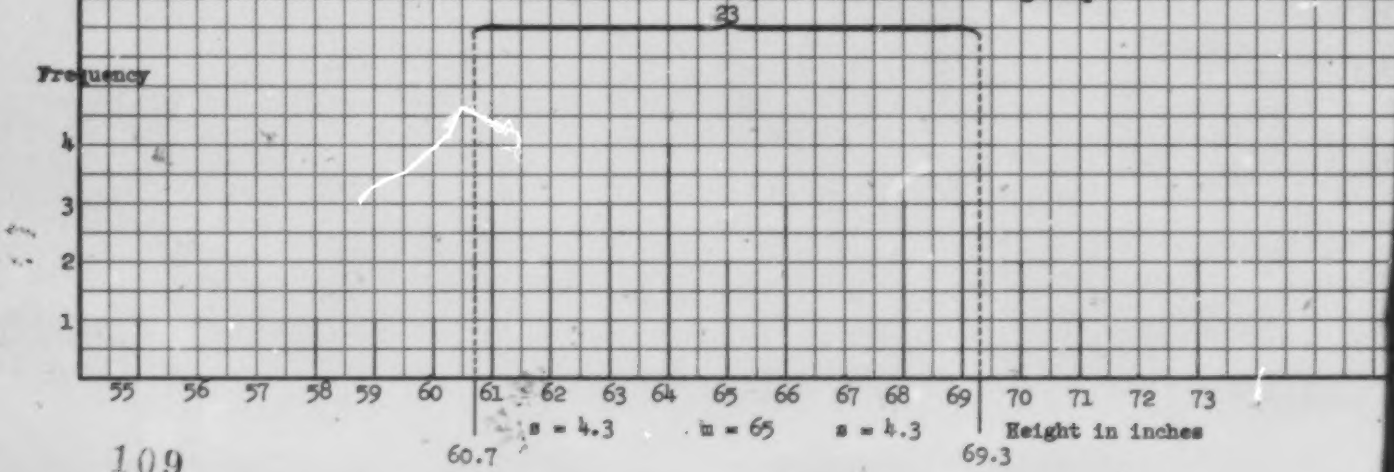
You may wish to examine other collections of data. Here are some suggestions:

1. Obtain from your teacher a set of test scores for your class.
2. Obtain the heights and weights for a class or grade in your school. (You may be able to secure this information from the Physical Education or Health Department.)
3. Find the number of hours each member of your class watched television last week.
4. Ask several members of your class to use a meter stick to measure the length of the teacher's desk to the nearest centimeter. The results will vary a little. Tabulate the result for each member of the class.
5. Many science projects involve collections of numerical data. Your science teacher may be able to give you suggestions.

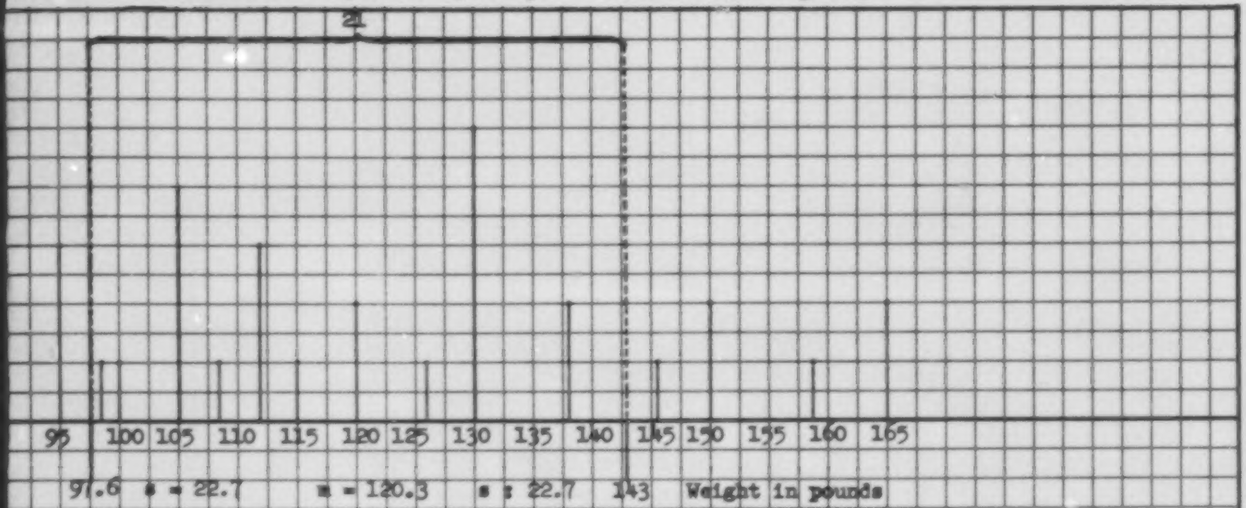
Graph of the Weights of 32 Boys -- Stanford Coaching Camp



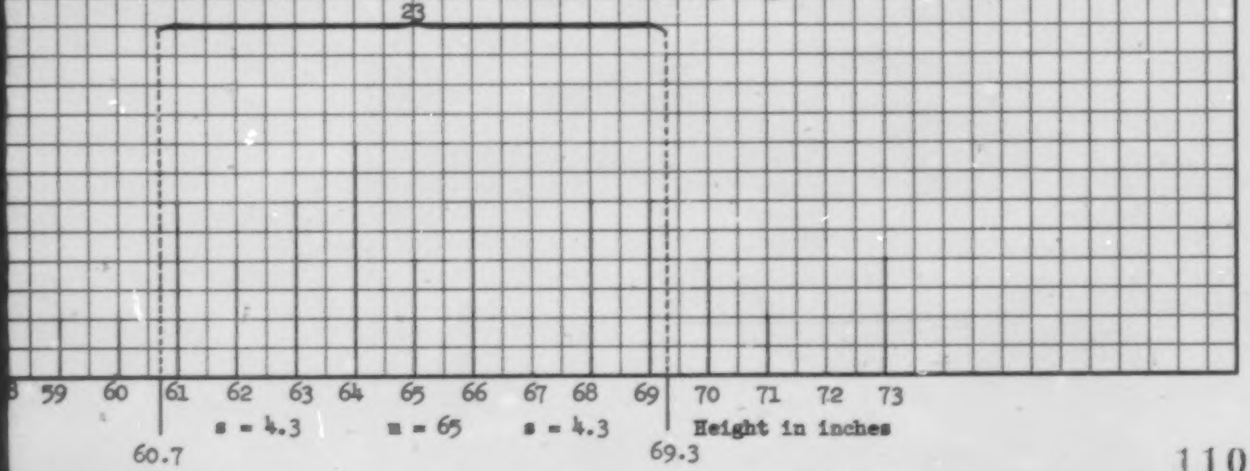
Graph of the Heights of 32 Boys -- Stanford Coaching Camp



Graph of the Weights of 32 Boys -- Stanford Coaching Camp



Graph of the Heights of 32 Boys -- Stanford Coaching Camp



10-8. Standard Deviation: Application

We have already discussed the standard deviation as a measure of dispersion for a frequency distribution. Naturally, we can also apply this idea to probability.

In Section 10-3, we considered tossing 3 coins. We made the following table:

Number heads	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

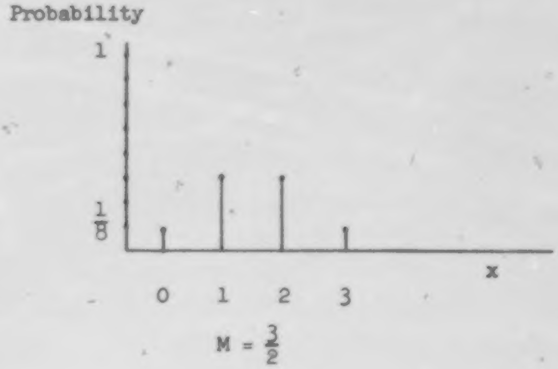
Table I

Such a table defines a probability distribution. (Observe that this probability distribution assigns to each of the numbers 0, 1, 2, 3 a probability.)

For this probability distribution, we have found that

$$M = \text{Expected value} = \frac{3}{2} .$$

We can draw a bar graph for this distribution. (You may recall that we have already used bar graphs in connection with probability distributions in Section 9-5.)



We can also ask: What is the standard deviation for this distribution? It is natural to proceed as follows:

$$(0 - \tfrac{3}{2})^2(\tfrac{1}{8}) + (1 - \tfrac{3}{2})^2(\tfrac{3}{8}) + (2 - \tfrac{3}{2})^2(\tfrac{3}{8}) + (3 - \tfrac{3}{2})^2(\tfrac{1}{8}) = \tfrac{3}{4}$$

$$\text{Standard deviation} = \sqrt{\tfrac{3}{4}}$$

$$= \tfrac{1}{2}\sqrt{3} , \text{ or approximately } .87 .$$

To find the standard deviation, then, we form the "weighted" average of the values of $(x - M)^2$, where the values of x are 0, 1, 2, 3.

Test your understanding by completing the following items. Refer to the work above as necessary.

$\frac{1}{8}$ 1. The value 0 (corresponding to no heads) has probability ______.

$\frac{3}{2}$ 2. M, the expected value, is ______.

$(0 - \frac{3}{2})^2$ 3. For $x = 0$, we have: $(x - M)^2 = \underline{(\quad - \quad)}^2$,

$(-\frac{3}{2})^2$ or $\underline{(\quad)}^2$.

$\frac{9}{4}$ 4. We know, of course, that $(-\frac{3}{2})^2$ = ______.

4 5. In computing the standard deviation, we used a sum of ______ terms, corresponding to the 4 possible values of the number of heads.

1

$\frac{3}{8}$ 6. The second term of this sum is $(1 - \frac{3}{2})^2(\frac{3}{8})$. It corresponds to the value ______ of x, which has probability ______.

In this chapter we have studied two kinds of situations.

We have seen examples in which data found by observation can be recorded in a frequency table.

For example, the authors actually tossed 3 coins 40 times, recording the number of heads. These were our results:

Number of heads	0	1	2	3
Frequency f	4	16	17	3
$\frac{f}{40}$	$\frac{4}{40}$	$\frac{16}{40}$	$\frac{17}{40}$	$\frac{3}{40}$

Table II

On the other hand, we have considered probability distributions, such as the one at the beginning of this section.

In both situations, we can compute the mean and the standard deviation.

Exercise. (Answers on page 338.)

1. Find the mean and the standard deviation for our coin-tossing results.

Let us look again at Table I and Table II. Compare the numerical results that we have obtained from them.

$\frac{1}{8}$ 7. If we think of a very large number of 3-coin tosses, say, 1,000,000, we would get no heads about ______

$\frac{3}{8}$ of the time, 1 head about ______ of the time, etc.

$\frac{3}{2}$ 8. For this situation, the average number of heads would be ______.

Table I, showing the probability distribution, summarizes results we would expect to find approximated if we repeated the experiment "toss three coins" a large number of times.

Imagine now that someone takes 1,000,000 slips of paper. This patient someone tosses 3 coins 1,000,000 times, recording one toss on each slip of paper. The slips are placed in a container, and they are well mixed. 40 are then selected. These 40 are a sample of the tosses. We can think of our experiment of tossing 3 coins 40 times as simulating (fitting the same rules as) the sampling process.

9. If we look at our second table, we observe that for it

1.48 the mean is ______ and the standard deviation

.77 is ______.

Our sample of 40 tosses gives fairly good estimates of the results for 1,000,000 trials.

Though no one, of course, would really toss 3 coins 1,000,000 times, the ideas that we are working with are extremely useful in practical situations.

Often we want to know the characteristics of a large group. Manufacturers of boys' clothing, for example, need to know about the heights and weights of boys in order to make plans about the number of articles of various sizes that they should produce. They need to know the average (mean) size of boys of various ages. They also need to know how the sizes spread out from the mean. Their information is often gained by studying samples of the general population. Opinion pollsters, market analysts, and other individuals use samples to predict the behavior of large groups.

Exercises. (Answers on page 338.)

2. A die is tossed. We know that the expected value of the number of points thrown is 3.5. Find the standard deviation.
3. In Exercise 2, Section 10-2, you found the average for 25 throws of a die. Compute the standard deviation. Compare with your results for Exercise 2 above.

10-9. A Formula for Standard Deviation

In this chapter we have obtained a number of results relating to the throw of a single die. We found:

Expected value of number thrown $= \frac{7}{2}$ (Item 3, Section 10-3)

Expected value, square of number thrown $= \frac{91}{6}$ (Exercise 4, Section 10-4)

Standard deviation, number thrown $= \sqrt{\frac{35}{12}}$ (Exercise 2, Section 10-8)

$\frac{49}{4}$

$\frac{147}{12}$

$\frac{35}{12}$

1. Observe that $(\frac{7}{2})^2 = \frac{\square}{\square}$.
2. Moreover, $\frac{91}{6} - \frac{49}{4} = \frac{182}{12} - \frac{\square}{12}$

   $= \underline{\qquad}$.

We see:

$$\frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} .$$

When we see a simple relationship in one example, we hope that it holds generally. Does this one?

First, we had better state the formula more generally. Suppose we have an experiment, the outcomes of which can be specified in terms of certain numbers. (In the example, the numbers are 1, 2, 3, 4, 5, 6.) We shall write:

$E(N)$ for the expected value of the number resulting from the experiment;

$E(N^2)$ for the expected value of the square.

Then we have--at least in our example:

$$E(N^2) - (E(N))^2 = (\text{standard deviation})^2.$$

The formula always holds. The proof uses some simple ideas from algebra, and if you are taking algebra, you might like to try it.

We will give a proof for a simple situation in order to save writing. Let us suppose, then, that we have a table as follows:

Number associated with experiment	x	y	z
Probability	P(x)	P(y)	P(z)

Since every outcome of the experiment must result in x or y or z, we have:

1 — 3. $P(x) + P(y) + P(z) =$ ______.

4. For the expected value M, we have:

zP(z) — $M = E(N) = xP(x) + yP(y) +$ ______.

$z^2P(z)$ — 5. Also, $E(N^2) = x^2P(x) + y^2P(y) +$ ______.

6. $(\text{standard deviation})^2 = (x - M)^2P(x) + (y - M)^2P(y)$

$(z - M)^2P(z)$ — $+$ ______.

$x^2 - 2Mx + M^2$

$M^2P(x)$

7. We know, of course, that:

$$(x - M)^2 = \underline{x^2 - \qquad}.$$

$$(x - M)^2P(x) = x^2P(x) - 2xMP(x) + \underline{\qquad}.$$

Thus, we can multiply out in Item 6. Regrouping the terms in the result, we have:

$$\begin{aligned}(\text{standard deviation})^2 &= (x^2P(x) + y^2P(y) + z^2P(z)) \\ &\quad - 2M(xP(x) + yP(y) + zP(z)) \\ &\quad + M^2(P(x) + P(y) + P(z)).\end{aligned}$$

Look back at Items 5, 4, and 3, in that order. Using them, our formula becomes:

M; 1

M^2

8. $$\begin{aligned}(\text{standard deviation})^2 &= E(N^2) - 2M(\underline{\quad}) + M^2(\underline{\quad}) \\ &= E(N^2) - 2M^2 + M^2 \\ &= E(N^2) - \underline{\qquad}.\end{aligned}$$

Since $M = E(N)$, we have:

$$(\text{standard deviation})^2 = E(N^2) - (E(N))^2.$$

If you have completed Chapter 9, the following part of this section will show you something about the standard deviation in the case of Bernoulli trials.

Exercises. (Answers on page 339.)

1. In Section 10-9 we found that for the number of heads on a toss of 3 coins we have:

$$M = \frac{3}{2}$$

$$\text{Standard deviation} = \frac{1}{2}\sqrt{3}$$

Verify this result, using the formula above.

2. Find the expected value and standard deviation of the number of heads if 4 coins are tossed.

3. Find the expected value and standard deviation of the number of 1's if a die is tossed 3 times.

The results of these exercises again suggest certain generalizations. For example, we had:

For 3 coins, standard deviation = $\frac{1}{2}\sqrt{3}$

For [illegible] coins, standard deviation = 1 = $\frac{1}{2}\sqrt{4}$

Guess $\frac{1}{2}\sqrt{5}$

9. For 5 coins, standard deviation = ______.

Moreover, we had:

For 3 coins, standard deviation of number of heads = $\frac{1}{2}\sqrt{3}$

For 3 dice, standard deviation of number of 1's = $\frac{1}{6}\sqrt{15}$.

$\sqrt{3}$

$\frac{1}{6}$

$\frac{5}{6}$

10. We recognize, of course, that $\sqrt{15} = \sqrt{5} \cdot \sqrt{\quad}$ ______.

11. We also note that for throwing a die, the probability of 1 on a throw is ______, and the probability of not getting 1 on a throw is ______.

It begins to look as though Exercises 1, 2, 3 all illustrate some general rule. Indeed, they do. For Bernoulli trials, we have a simple formula for the standard deviation of the number of successes.

If n is the number of trials, p the probability of success for each trial, and q the probability of failure for each trial (so that $q = 1 - p$), then:

standard deviation of number of trials = $\sqrt{npq}$.

For $n = 3$, $p = q = \frac{1}{2}$ (Exercise 1)

standard deviation = $\sqrt{3 \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{2}\sqrt{3}$.

For $n = 3$, $p = \frac{1}{6}$, and $q = \frac{5}{6}$ (Exercise 3)

standard deviation = $\sqrt{3 \cdot \frac{1}{6} \cdot \frac{5}{6}} = \frac{1}{6}\sqrt{15}$

For $n = 4$, $p = q = \frac{1}{2}$ (Exercise 2)

standard deviation = $\sqrt{4 \cdot \frac{1}{2} \cdot \frac{1}{2}} = 1$

Exercise. (Answers on page 340.)

4. Prove: For $n = 3$ and any p, q, show that the standard deviation of the number of successes is $\sqrt{3pq}$.

†10-10. An Example: The World Series.

Here is an application of ideas from both Chapter 9 and 10. Do it only if you have completed the early sections of both.

A reporter who is going to cover the World Series wonders how long it will last. In an old book, he finds the following records of past series.

Year	American League	National League	Score
1925	Washington	Pittsburgh	4-3
1926	New York	St. Louis	4-3
1927	New York	Pittsburgh	4-0
1928	New York	St. Louis	4-0
1929	Philadelphia	Chicago	4-1
1930	Philadelphia	St. Louis	4-2
1931	Philadelphia	St. Louis	4-3
1932	New York	Chicago	4-0
1933	Washington	New York	4-1
1934	Detroit	St. Louis	4-3
1935	Detroit	Chicago	4-2
1936	New York	New York	4-2
1937	New York	New York	4-1
1938	New York	Chicago	4-0
1939	New York	Cincinatti	4-0
1940	Detroit	Cincinatti	4-3
1941	New York	Brooklyn	4-1
1942	New York	St. Louis	4-1
1943	New York	St. Louis	4-1
1944	St. Louis	St. Louis	4-2
1945	Detroit	Chicago	4-3
1946	Boston	St. Louis	4-3
1947	New York	Brooklyn	4-3
1948	Cleveland	Boston	4-2
1949	New York	Brooklyn	4-1
1950	New York	Philadelphia	4-0
1951	New York	New York	4-2
1952	New York	Brooklyn	4-3
1953	New York	Brooklyn	4-2
1954	Cleveland	New York	4-0
1955	New York	Brooklyn	4-3
1956	New York	Brooklyn	4-3
1957	New York	Milwaukee	4-3
1958	New York	Milwaukee	4-3
1959	Chicago	Los Angeles	4-2
1960	New York	Pittsburgh	4-3

He can find the average number of games and use it to estimate how long the current series will last.

A second reporter says, "I have no data about past experiences. I do know, however, that of the two teams playing this year, the Blue Sox and the Green Sox, the Blue Sox are the better team. In fact, the probability that the Blue Sox win any particular game of the series is $\frac{2}{3}$."

Exercises. (Answers on page 340.)

1. Using the first reporter's data, what is the average number of games?
2. Using the second reporter's probability estimate, what is the expected number of games for the World Series?

(Notice that the reporter assumes that the games are independent trials. The probabilities are the same for all the games.)

You might be interested in seeing what the results would have been if the second reporter had made a different estimate of the probability of the Blue Sox winning any particular game. Suppose, for example, the Blue Sox win every game with probability $\frac{3}{4}$. In other words, the teams are matched 3 to 1. Or suppose they are matched equally, 1 to 1, so that for each game the Blue Sox (and Green Sox) have probability $\frac{1}{2}$ of winning. The results for these cases, as well as that of Exercise 2, are summarized below.

Summary of the three World Series:

Probability that the Series ends in:	The teams are matched		
No. of games	1 to 1	2 to 1	3 to 1
4	0.12	0.21	0.32
5	0.24	0.29	0.33
6	0.32	0.27	0.22
7	0.32	0.22	0.13
The expected number of games	5.84	5.45	5.16

Chapter 11

BERTRAND'S BALLOT PROBLEM

11-1. Experiment

A certain class in which there were 25 students had an election between Arthur and Barry. When the votes were collected, they were read off as follows:

A A B A B A A B A B A A A B A A B B A B A A B B B

After the votes were read off, one student said, "I notice that Arthur was always in the lead. This is surprising."

A second student replied, "Not at all: The final vote was 14 to 11 in favor of Arthur. Since Arthur won, it is quite likely that he was always ahead."

Do you agree with the second student? Think about it. Then, see what you can find out by experimenting. Make 25 slips of paper, and mark 14 A and 11 B. Mix them well in a container. Draw them out one by one, recording your result as we have done above. If several students do this experiment, each should keep his own record. Save your record; you will need it again. Examine your results. Does A always lead?

An easy way to check is to think: a vote for A is 1. A vote for B is (-1). Using these values, add the votes as you read along the list. Thus, in the vote above you read:

1, 2, 1, 2, 1, 2, 3, 2, 3, 2, 3, 4, 5, 4, 5, 6, 5, 4, 5, 4, 5, 6, 5, 4, 3

Compare your results with ours. (Discussion of this experiment is on page 308.) After you have examined your results, and, perhaps, compared yours with those of other members of your class, consider again the comments of the two students. With which one do you agree? The following questions will help you to test whether you were correct.

1. If A always leads, then what must the first vote be? the second vote?
2. What is the probability that A leads after 2 votes are counted?
3. What is the probability that A leads after 3 votes are counted?
4. What can you conclude about the probability that A always leads?

11-2. Bertrand's Ballot Problem Stated

The French mathematician J. Bertrand (1822-1900) studied a problem about elections. The problem: If we know the final results of an election, can we find the probability that, as the votes are counted, the winner is always in the lead?

Thus, in our example, we would ask: What is the probability--if the vote was 14 to 11--that Arthur always led?

Whether or not A always leads depends on the way in which the 14 A's and 11 B's are arranged in a sequence. We will call each possible arrangement an "ordering".

The event "A always leads" is a subset of a certain set of outcomes. This set of outcomes consists of all the orderings in which the votes could have been read off.

1. In order to find the desired probability we could look for the total number of orderings and the number of orderings of the subset "A always ______".

leads

2. We would also need to know whether or not all these outcomes are ______ likely.

equally

If the ballots drawn from a container are all the same size and are folded the same way and mixed well, we can assume that all possible orderings are equally likely. We shall do so.

Now consider the question: How many possible equally likely orderings are there? A very great many!

If you have studied the chapter on Bernoulli trials (Chapter 9), you should be able to calculate the number of 25-letter words with 14 A's and 11 B's which are possible. This number is $\binom{25}{11}$ --the number of ways of choosing 11 positions for B out of 25 possible positions. The number turns out to be 4,457,400.

There are two surprising things about this problem. First, it has an easy answer. Second, it is not difficult to discover the answer and to see, at least in a general way, why it should be true.

Let us look for the answer. How shall we begin? Think about this; see whether our procedure in the next section is what you would have suggested.

11-3. A Simpler Case

When a problem looks difficult it is often helpful to try a similar but simpler case. We might begin by seeing what happens when the number of voters is smaller.

Suppose that there are only 5 voters, and that A wins by a vote of 4 to 1.

1. List in a column all the different 5-letter orderings with 4 A's and 1 B. Decide for each whether A always leads. Check your results with the list below. (Your list should contain the same entries as ours, though not necessarily in the same order.)

Ordering	Does A lead throughout?
A A A A B	Yes
A A A B A	Yes
A A B A A	Yes
A B A A A	No
B A A A A	No

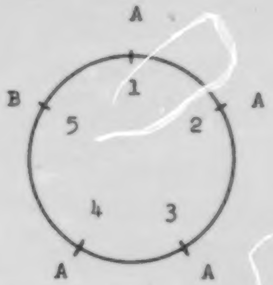
Note that we have $\binom{5}{1}$, or 5, possible orderings, corresponding to 5 choices of position for B.

B A A A A

2. If we count A as 1 and B as -1, we may count the score for each 5-letter word step by step. For which words does A not lead throughout? A B A A A and ______.

equally	3. There are 5 possible orderings if the score is 4 to 1; that is, 5 ______ likely outcomes.
3	4. Of these, ______ are in the event "A leads throughout".
$\frac{3}{5}$	5. The probability that A leads throughout is $\frac{\square}{\square}$.

Instead of writing a list it is helpful to enter the votes on a circle. Examine the circle below.



If we begin at 1 and go clockwise around the circle, we have A A A A B.

A A A B A	6. If we begin at 2 and go clockwise back to 2, we get ______.
	Note that each entry on our list of orderings corresponds to a different beginning point on the circle.
1, 2, 3	7. The beginning points that give an ordering for which A is always ahead are ______, ______, and ______.

Having dealt with 5 votes where the vote is 4 to 1, we might also consider a vote of 3 to 2.

Exercises. (Answers on page 344.)

1. How many possible orderings are there in this case?

2. Choose any single ordering for 3 A's and 2 B's. Locate the A's and B's on a circle, so that if you begin with position 1 and move clockwise around the circle, you get the ordering you chose. List all the possible orderings you get by choosing different starting points and going clockwise around this circle. (You should have 5 in all, including the one you began with.)

3. For which beginning positions on your circle do you get orderings in which A always leads?

4. Does your list in Exercise 2 include all the possible orderings of 3 A's and 2 B's ? If not, select an ordering not on your list. Again draw a circle, labeling it for this ordering. Again list all the outcomes for this circle. (Always going around clockwise, of course.)

5. For how many beginning positions on your second does A always lead?

6. What is the probability that A always leads for a 3 - 2 vote?

7. If the vote is 5 - 0, what is the probability that A always leads?

8. Let us summarize our results. Prepare a table, showing for 4 to 1 votes and for 3 to 2 votes the following information:

    P(A always leads)
    Number outcomes
    Number circles
    Number starting positions on each circle for which A always leads.

    Compare your table with the one below.

Vote	P(A always leads)	Number outcomes	Number of circles used	Number "A always leads" starting positions for a circle
4 to 1	$\frac{3}{5}$	$\binom{5}{1}$, or 5	1	3
3 to 2	$\frac{1}{5}$	$\binom{5}{2}$, or 10	2	1

Examine carefully the first and second columns of this table. Then, think about the following questions.

4 9. For the first row, A got ______ votes and B
1 got ______.

$\frac{3}{5}$ 10. In this case the probability of "A always leads" is ______.

How do you obtain 3 from the numbers in Item 9? How do you obtain 5 ?

Can you see a pattern? Look at the numbers 3, 2, and $\frac{1}{5}$ in the second row. Do they fit your pattern?

By now, you may have recognized a rule which seems to apply. Let:

x = number of votes of A; and
y = number of votes of B.

11. Complete the following table. Compare with the completed table below.

Vote	P(A always leads)	x	y	x-y	x+y	$\frac{x-y}{x+y}$
4 to 1	$\frac{3}{5}$	4	1			
3 to 2	$\frac{2}{5}$					

Vote	P(A always leads)	x	y	x-y	x+y	$\frac{x-y}{x+y}$
4 to 1	$\frac{3}{5}$	4	1	3	5	$\frac{3}{5}$
3 to 2	$\frac{2}{5}$	3	2	1	5	$\frac{1}{5}$

Compare the second and last columns. It appears that:

$$P(\text{A always leads}) = \frac{x - y}{x + y}.$$

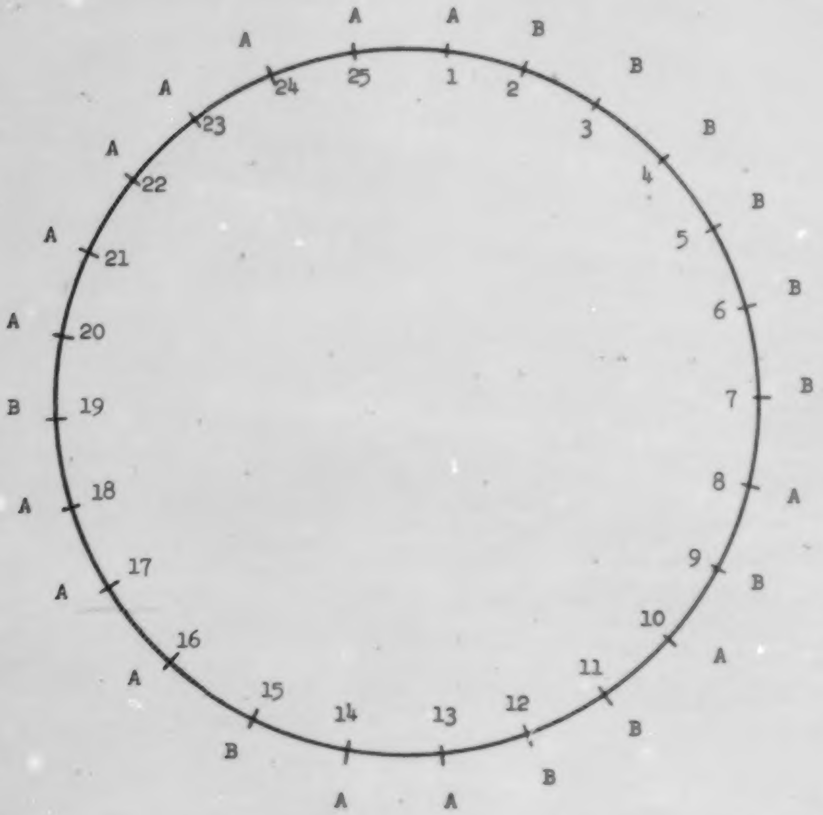
At least, this formula holds for votes of 4 to 1 and 3 to 2.

Exercises. (Answers on page 344.)

8. Does the formula hold if the vote is 5 to 0 ?

9. Does the formula hold if there are 3 votes and the vote is 2 to 1 ?

Here is the result of our experiment for 25 votes. We have arranged our ordering on a circle.


A 25
A 1
B 2
B 3
B 4
B 5
B 6
B 7
A 8
B 9
A 10
B 11
B 12
A 13
A 14
B 15
A 16
A 17
A 18
B 19
A 20
A 21
A 22
A 23
A 24

Study this circle.

12. If you begin counting at position 1, then A
does not ______ always lead. (In fact, the vote is
(does, does not)
tied at the second vote.)

13. Suppose, however, you begin at position 13 and go
clockwise. Does A always lead for this ordering?
yes ______.

14. The other orderings on this circle for which A
16, 17 always leads begin at ______ and ______.

Exercise. (Answer on page 345.)

10. Refer to your record for 25 votes (Experiment 11-1). Arrange the 25 votes on a circle. For how many starting positions does A always lead?

11-4. Examining Our Result

Your experiences in Section 11-3 have suggested that: If

$$x = \text{number of votes for } A,$$
$$y = \text{number of votes for } B,$$

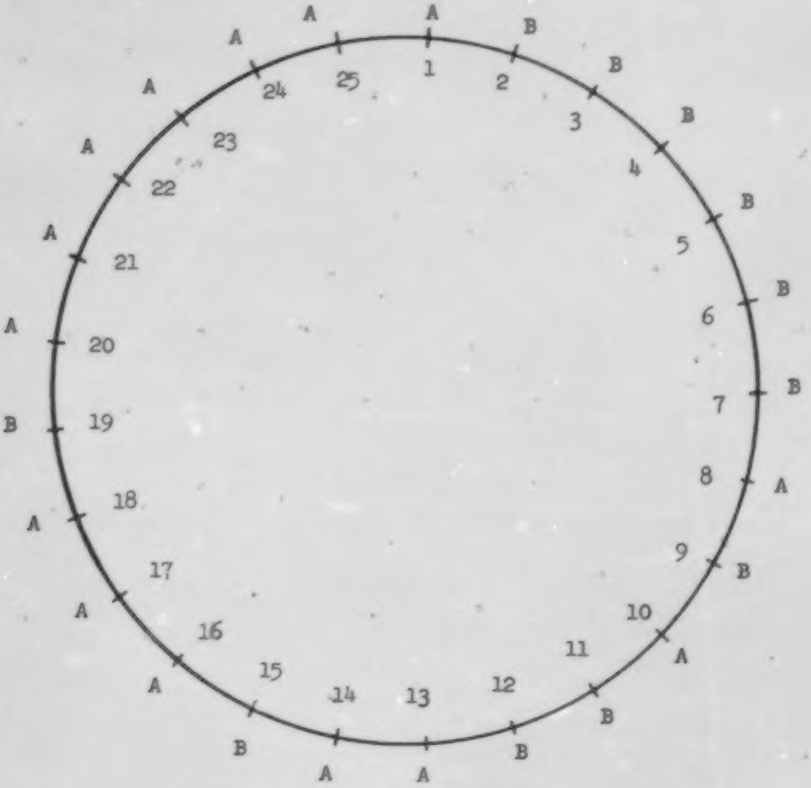
and if

$$x > y \quad (\text{that is, } A \text{ wins})$$

then,

$$P(\text{A always leads}) = \frac{x - y}{x + y}.$$

In order better to understand why this is true, we will examine the results we got for 25 votes. We entered our ordering on a circle.


A
1
B
2
B
3
B
4
B
5
B
6
B
7
A
8
B
9
A
10
B
11
B
12
A
13
A
14
B
15
A
16
A
17
A
18
B
19
A
20
A
21
A
22
A
23
A
24
A
25

B

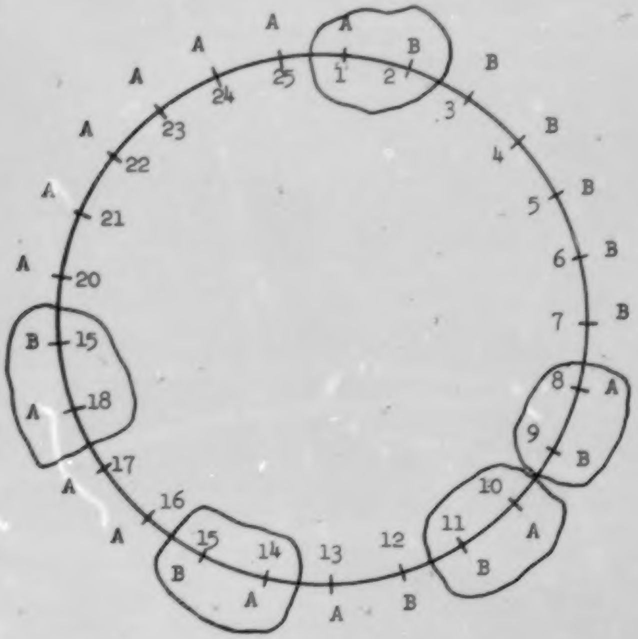
0

1. Position 1 starts with A followed by ______.
   But 1 + (-1) = ______, so A has lost the lead at the second vote.

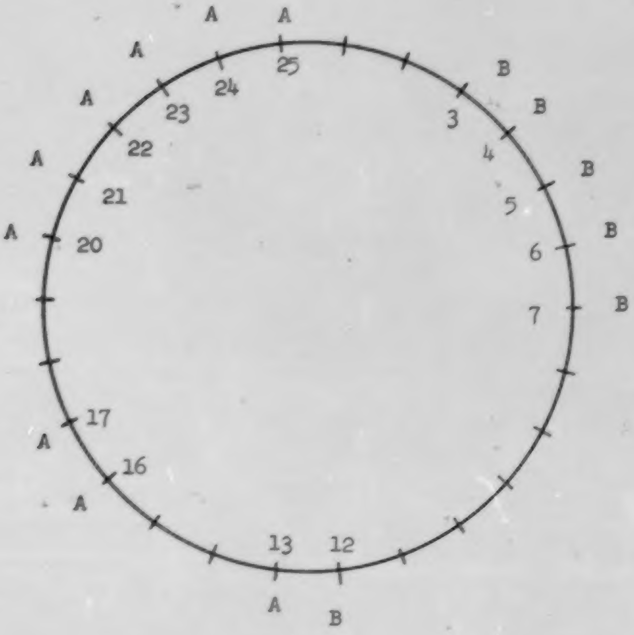
14, 18

2. We have the same situation, A followed by B, at starting positions 8, 10, ______, and ______.

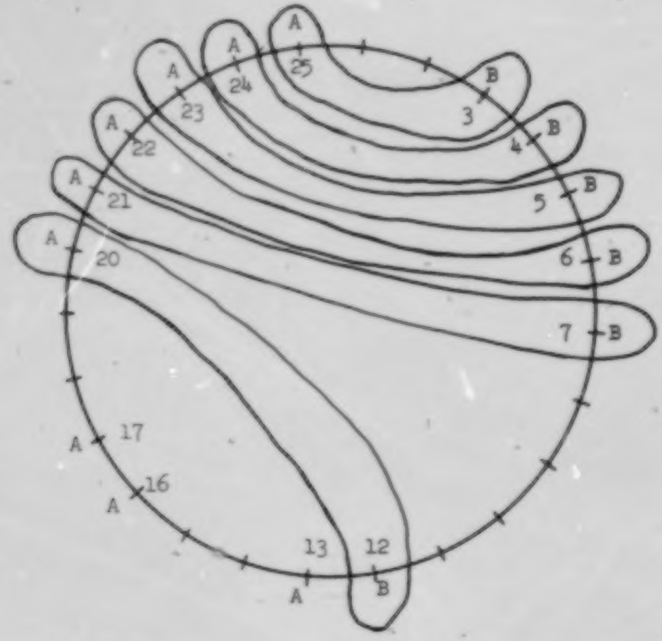
In the circle below, we have indicated the pairings of Item 1 and 2.


A
A
A
B
B
1
2
25
24
A
A
3
B
4
23
A
22
B
5
A
21
B
6
A
20
B
7
B
15
8
A
A
18
9
B
17
A
10
16
A
A
15
14
13
12
11
B
B
A
A
B

Take these pairs out of the circle and you have:


A
A
A
25
B
24
A
23
3
B
22
4
A
21
B
5
A
20
6
B
7
B
17
A
16
A
13
12
A
B

On the above circle do you have an A followed by a B if you move around in a clockwise direction? Yes--the A at 25 and the B at 3. Eliminate these. Continue around again in a clockwise direction. You come to the A at 24. It is followed by the B at 4. Eliminate them. On the next round you eliminate 23 and 5, then 22 and 6. What next? Of course, 21 and 7, and then 20 and 12. See the diagram below.



Which positions have not been eliminated? 13, 16, and 17. But we have found that positions 13, 16, and 17 are exactly the positions giving orderings in which A always leads.

Does this give the clue for our generalization? If we place our ballots around a circle in the order in which they are called, and if we eliminate the pairs of winner and loser that follow, in that order, as we go around the circle clockwise, over and over, we are bound to be left with the positions on the circle from which the winner always leads. If you don't believe this, try some more cases and think a little harder.

But how many places will be left each time? Exactly the number by which the winner's votes exceeds the loser's: exactly x - y.

11-5. Summary

In Section 11-3 we were led to the guess:

> If x = number of votes for A, and y = number of votes for B, then:
>
> $$P(\text{A always leads}) = \frac{x - y}{x + y}.$$

(Notice, we are always supposing A wins, so that $x > y$.)

In Section 11-4 we got a clue as to why this formula holds. We saw that if we consider only the orderings that can be indicated on a single circle, then we have:

> the number of orderings where A always leads = $x - y$.

Referring again to our case in which the vote is 3 to 2, we noted that 2 circles are needed to give all the outcomes. For each, there are 5 orderings. We have:

> Number of orderings on first circle where A always leads = 1 ;
>
> number of orderings on second circle where A always leads = 1.

$$\frac{\text{total number of orderings where A always leads}}{\text{total number of orderings}} = \frac{1 + 1}{5 + 5} = \frac{1}{5}$$

You might make a guess about what happens if you have 7 votes, with 5 for A and 2 for B.

$\binom{7}{5}$ or $\binom{7}{2}$, 21	1. There are in all $\binom{\ }{\ }$, or ______ orderings in this case.
3	2. It appears reasonable that you would need ______ (how many) circles to list all the possible orderings. (Each circle would give 7 of them.)
5 - 2, or 3	3. For each, it appears that there would be ______ (how many) starting positions for which A would always lead. (The reasoning would be exactly like that in Section 11-4.)
$\frac{3}{7}$	4. It appears reasonable, then, to suppose that in this case: $P(\text{A always leads}) = \frac{3 + 3 + 3}{7 + 7 + 7} =$ ______.

Once again it seems that:

$$P(\text{A always leads}) = \frac{x - y}{x + y}.$$

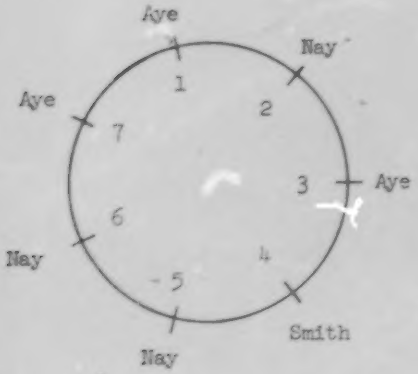
As a matter of fact, the formula does hold in all cases. The statements in Items 1 to 4 are true. (You could, of course, check them if you wish.)

However, the reasoning outlined in Items 1 to 4, though valid for the 7-vote example, is not always complete. If you would like to think a little more about this problem, the next section will help you understand it more fully.

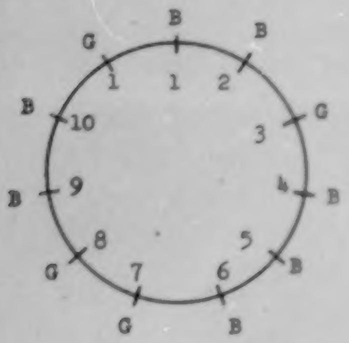
Exercises. (Answers on page 345.)

1. A committee of 10, seated around a table, will vote 7 ayes and 3 nays. What is the probability that, as the chairman calls on each man for his vote, the ayes lead all through the count?

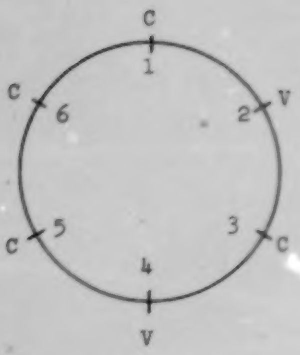
2. Senator Slattery is holding a committee meeting. The committee is seated around the table as in the figure. Slattery knows how everyone will vote except Senator Smith. He knows Senator Smith well enough to know that if the aye's are ahead when Slattery calls for Smith's vote, Smith will vote "aye". Slattery wants all the aye votes he can get. How shall Slattery start the count around the table to be sure the aye's are ahead when he calls on Senator Smith?



3. At an informal party the boys and girls seated themselves around the table as in the figure to the left. When supper was finished, the hostess called, "Boys, each of you bring the girl on your left to the patio for mixed ping pong." Which boys ended up without partners?



4. The Happy family is sitting around the dinner table. Mrs. Happy is passing the cupcakes. There are 3 chocolate and 3 vanilla. How can the cupcakes be passed clockwise so that Daddy, in seat 1, will be sure to get chocolate? The letters around the circle indicate the preference of each member of the family.



5. In a 24-event competition, team A won 16 events. What is the probability that team A led throughout the scoring?

6. The World Series stands 4 to 3. What is the probability that the team which has won 4 games was always in the lead?

7. Gonzales won a tennis set of 10 games. The game score was 6-4. What is the probability that he led in the game score throughout the set?

T8. Tilden won a 42-game set of tennis 22-20. What was the probability that he led all the way? Careful.

9. A king is attacked by 9 B's and defended by 13 A's. These 22 men reach him in random order. The king is lost unless the number of defenders who have reached him always exceeds the number of attackers. What is the probability that the king survives?

T10. A football player catches the ball and starts running toward the goal line for a touchdown. Between him and the goal line are 3 members of his team and 2 of the opponents' team. He'll be downed if the number of opponents ever exceeds the number of members of his own team who reach him. What is the probability that he will reach the goal?

11. The papers the day after election announce that Joe Doe won by a landslide. The vote is 2 to 1 in his favor. What is the probability that he led all through the count?

T11-6. Some Further Considerations

In this chapter we have used circles to help us list possible orderings.

5

25

1. For 5 votes we used a circle with 5 positions, and this circle gave us ______ possible orderings.

2. Likewise, our 25 vote circle gave us ______ possible orderings.

Can we always suppose things are quite this simple?

One more example will enable you to answer the last question.

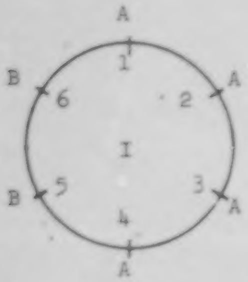
Exercises. (Answers on page 345.)

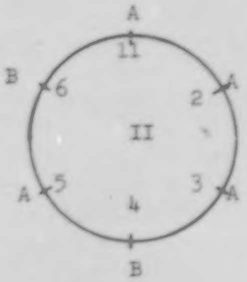
1. Consider an election with 6 votes--4 for A and 2 for B. How many possible orderings are there?

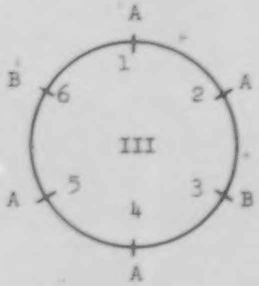
2. Choose one possible ordering, draw and label a circle for this ordering, and list all the different orderings (including the one you choose first)

for the circle. Select an ordering that does not appear on your list. Draw a circle for it. Continue, drawing circles and listing outcomes until you have all the possible outcomes. Work carefully.

In Exercise 2 above, you drew and labeled 3 circles. Though your circles were probably not exactly like ours, we have seen that they are essentially the same. Here are the circles.







	You have found:
6	3. From Circle I, you get ______ different possible outcomes.
2	4. Of these, ______ are in the event "A always leads".
1, 2	5. They correspond to starting positions _____ and _____.
	Suppose you have held an election, and you are told that the outcome is one of the 6 for Circle I.
$\frac{2}{6}$ or $\frac{1}{3}$	6. Given this information, the probability that A always leads is ______.

Let:

E = event "A always leads",
I = event "ordering in Circle I",
II = event "ordering in Circle II",
III = event "ordering in Circle III".

$\frac{1}{3}$

7. We can rewrite Item 6 as:

$$P(E|I) = \_\_\_\_\_\_.$$

conditional

8. Note that $P(E|I)$ is a ______ probability.

$\frac{1}{3}$

9. In a similar way,

$$P(E|II) = \_\_\_\_\_\_.$$

Now let us consider Circle III.

3

10. We saw in Exercise 2 that Circle III has only ______ different outcomes.

4

11. For example, starting positions 1 and ______ lead to exactly the same ordering.

1

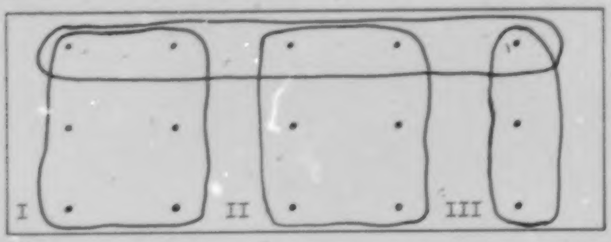
12. The starting positions 1, 2, 3 lead to different orderings. Of these, only the ordering beginning with position ______ is in event E.

$\frac{1}{3}$

13. Once more we have found a conditional probability:

$$P(E|III) = \_\_\_\_\_\_.$$

We may represent the 15 possible outcomes with the following diagram:



$E \cap III$

14. E can be regarded as the union of 3 sets-- $E \cap I$, $E \cap II$, and ______.

disjoint

15. These 3 sets are mutually ______. Hence, $P(E) = P(E \cap I) + P(E \cap II) + P(E \cap III)$.

At this point, we can apply our knowledge of conditional probability.

16. $P(E \cap I) = P(I) \cdot P(E|I)$. (See Chapter 7)

$\frac{1}{3}$ 17. Hence, $P(E \cap I) = P(I) \cdot \underset{\text{(number)}}{\underline{\qquad}}$. (Item 7)

$P(II) \cdot \frac{1}{3}$ 18. Similarly, $P(E \cap II) = \underline{\qquad \cdot \frac{1}{3}}$.

Also, $P(E \cap III) = P(III) \cdot \frac{1}{3}$.

We can substitute these results in Item 15.

19. We have:

$P(III) \cdot \frac{1}{3}$

$$P(E) = P(I) \cdot \frac{1}{3} + P(II) \cdot \frac{1}{3} + \underline{\qquad}$$

$$= (P(I) + P(II) + P(III)) \cdot \frac{1}{3}.$$

$P(I)$ is the probability the ordering can be read from Circle I. Similarly, $P(II)$, $P(III)$ are probabilities that the ordering can be read from II, III.

1 20. Hence, $P(I) + P(II) + P(III) = \underline{\qquad}$.

21. Thus, from Item 19,

$$P(E) = \frac{1}{3} \cdot 1 = \underline{\qquad}.$$

The reasoning illustrates the principle embodied in Bayes' formula.

In this example the vote was 4 to 2. Not all our circles gave 6 different outcomes. The reasoning of Section 11-4 applies to each circle. Moreover, the conditional probabilities associated with the different circles combine to give precisely the earlier result. More complicated cases give rise to more circles, but still to the same final results,

$$P(\text{A always leads}) = \frac{x - y}{x + y} \quad (x > y).$$

Chapter 12

MARKOV CHAINS*

12-1. The Careless Cook

A certain cook can prepare two cereals, Lumpies and Soggies, but sometimes she burns them. In fact, when she cooks Lumpies, her probability of burning it is .1. When she cooks Soggies, however, her probability of burning it is .4. Whenever she burns Lumpies, then she cooks Soggies the next day. However, she really doesn't like Soggies very well, even when it isn't burned. Consequently, after cooking it one day, she always goes back to Lumpies. (You may recall having met this careless cook in Section 8-2, Exercises 14 and 15).

You would suppose that the best advice you could give this cook would be to learn how to make pancakes. But, as you may have guessed, we will turn this situation into a problem in probability. As you have already seen, problems about probability involving dice, coins, and spinners can be translated into problems of very real importance in other areas. In the same way, the story of this rather incompetent cook illustrates general principles that are often used. These principles have to do with processes that go on for many steps.

Experiment (Discussion is on page 309.)

Here is an experiment that illustrates what the careless cook does about the cereal. We will suppose she cooks Lumpies on Monday, January 1. Put 9 white marbles and one red marble in a jar. In another jar, put 4 red and 6 white marbles. Label the first jar L and the second jar S. Make a record sheet as follows:

Day	Cereal cooked	Burned Yes/No
Mon., January 1	L	
Tues.		
Wed.		
Thurs.		
Fri.		
Sat.		

---

[*] Named after the mathematician A. A. Markov (1856-1922).

Now draw a marble from the L jar. If it is white, record "not burned" as the result for Monday. If it is red, record "burned". Put the marble back. If your result for Monday was "not burned", write L for the cereal cooked on Tuesday. If it was "burned", write S for Tuesday. In this latter case, draw a marble from the S jar. If it is red, mark "burned" for the Tuesday cereal. If it is white, mark "not burned".

If you wrote L for Tuesday, repeat in exactly the same way. If you wrote S for Tuesday, write L for Wednesday. (Remember, she never cooks S more than one day in a row.) Then repeat in exactly the same way.

1. Complete the record until you have recorded two days' cereal after your first S occurred.
2. About how often does the family eat Lumpies? About how often does it eat burned cereal? (Remember that when she cooks Soggies, her probability of burning it is .4.) Make your own estimates before you go on.

If the cook begins with the Lumpies, it is likely that for a few days all will go well. (The probability of burning the Lumpies on Monday is only .1.) Sooner or later, however, she will burn the Lumpies and change to Soggies for a day. Then she will change back, of course.

Suppose she goes on and on in this way until she has cooked L 1000 times in all.

100 — 1. We would expect that she would have burned L approximately _______ times.

100 — 2. Hence, we suppose she would have cooked S roughly _______ times.

1100 — 3. And in all there would have been _______ breakfasts. (Hint: The sum of the number of days she cooked L and the number she cooked S.)

4. We can now estimate how often the family is served Lumpies. As an approximation we have:

$\frac{1000}{1100}$, or $\frac{10}{11}$ — $$\frac{\text{number of times she cooks L}}{\text{number of breakfasts}} = \frac{\square}{\square}.$$

The cook serves Lumpies about $\frac{10}{11}$ of the time. How often does the family eat burned cereal?

100

5. Out of the 1000 times she cooks L, the cook burns it approximately ______ times.

40

6. Out of the 100 times she cooks S, she burns it approximately ______ times.

140

7. In 1100 days, we would expect that the family eats burned cereal about ______ days.

$\frac{140}{1100}$, or $\frac{7}{55}$

8. Approximately, we have

$$\frac{\text{number of times she burns the cereal}}{\text{number of days}} = \frac{\square}{\square}.$$

It appears that the family gets burned cereal about $\frac{1}{8}$ of the time.

12-2. More about the Careless Cook

Let us see how we can analyze the situation a little more carefully. We have supposed that the cook prepares Lumpies on Monday, January 1; that is, the probability that she cooks Lumpies on that day is 1.

In Exercises 14 and 15, Section 8-2, we used a tree to compute certain probabilities associated with the cook's activities. In making the tree, we used the following information given to us by the problem.

.1

1. P(burns L|cooks L) = ______.

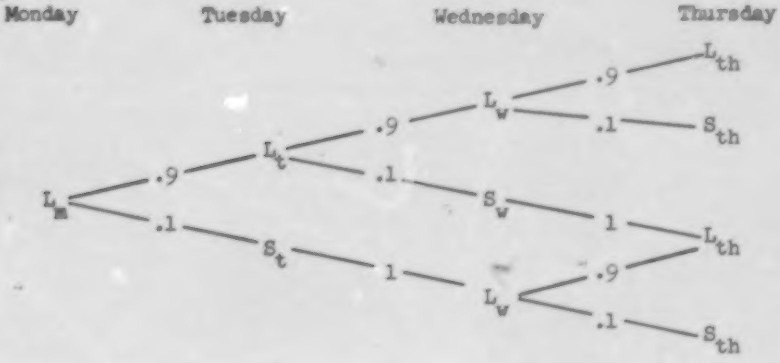
.1

2. Hence, P(cooks S on a given day|cooked L on preceding day) = ______.

1

3. P(cooks L on a given day|cooked S on preceding day) = ______.

4. Using this information, make a tree showing 4 days cereal-cooking possibilities. Check your tree with the one shown below.


Monday
Tuesday
Wednesday
Thursday
L_m
.9
.1
L_t
S_t
.9
.1
1
L_w
S_w
L_w
.9
.1
1
.9
.1
L_{th}
S_{th}
L_{th}
S_{th}

We have used L_t for "Lumpies Tuesday", S_w for "Soggies Wednesday", etc.

From the tree, let us compute some probabilities.

1	5. $P(L_m)$ = ______.
.9	6. $P(L_t)$ = ______.
.9 × .9	7. $P(L_w)$ = (____ × ____) + (.1 × 1).
.91	Hence, $P(L_w)$ = ______.
.9 × .1, .09	8. $P(S_w)$ = ____ × ____ = ____.
	9. We note, of course, that
1	$P(L_w) + P(S_w)$ = ______.

	Here is a short cut for computing $P(L_{th})$.
.1; .091	10. $P(S_{th})$ = ______. $P(L_w)$ = ______.
1 - .091, .909	11. Hence, $P(L_{th})$ = ___ - ___ = ______.

12. Of course, you could also find $P(L_{th})$ directly:

$$P(L_{th}) = (.9 \times .9 \times .9) + (.9 \times .1 \times 1) + (\_\_\_\_\_) = \_\_\_\_\_.$$

$.1 \times 1 \times .9$, or .09

.909

The results we have obtained can be summarized as follows. The probability of cooking cereal L is:

1	for Monday
.9	for Tuesday
.91	for Wednesday
.909	for Thursday.

What would happen if you went on and on computing probabilities for successive days? Let us think a little more about it.

Exercises. (Answers on page 346.)

1. Compute the probability of cooking Lumpies on Friday. Use the short-cut method.
2. By computation or by a clever guess, find the probabilities of L for Saturday and for Sunday.

13. Guess: The probability of cooking L on Monday, January 8, is ______.

.9090909

Look back at Item 4, Section 12-1.

14. Write $\frac{10}{11}$ as a decimal: $\frac{10}{11} = .\_\_\_\_\_$.

.909090..., or $.\overline{90}$

Our work suggests that the probability of cooking Lumpies changes from day to day, but as time goes on it gets closer and closer to $\frac{10}{11}$.

Exercises. (Answers on page 346.)

3. What is the probability of burned cereal on Monday?
4. Write $\frac{7}{55}$ as a decimal.

12-3. The Careless Cook Uses a New Plan

The careless cook finds that she has grown to like Soggies, so she changes her plan of operation. She begins, on Monday, January 1, another year, by cooking Lumpies. Again, she cooks Lumpies until she burns it, and then changes to Soggies. Now, however, she cooks Soggies until she burns that, and then changes back to Lumpies again. Unfortunately, in all this time her cereal-cooking has not improved. Her probability of burning L if she cooks it is .1, and her probability of burning S if she cooks it is .4. Exercises 16 and 17, Section 8-2, refer to this situation.

Experiment. (Discussion on page 310.)

1. Write a description of an experiment which would match the cook's behavior, assuming the cook begins with Lumpies. Record the experiment that would correspond to 30 days of cooking cereal, beginning with L on the first day.

2. Would you expect that

   (a) the probability of cooking L would not be the same for each day?

   (b) cooking L would continue to be more likely than cooking S ?

3. If she cooks cereal every morning for several months, is one kind of cereal burned oftener than the other?

Again, we can estimate what happens if the cook goes on this way for a long time. Try to do this, using the method of Section 1. Use Items 1 to 7 for a help or a check.

100 1. If she cooks L for 1000 days, she will burn it approximately ______ times.

100 2. This means that she will change from L to S approximately ______ times.

100 3. Consequently, she will also change from S to L approximately ______ times, which means she burns S

100 approximately ______ times. (Refer to Question 3 in the experiment.)

.4

4. She burns S about ______ of the times she cooks it.

250

5. Hence, if she burned it about 100 times, she cooked it about ______ times. (100 = .4(250))

250

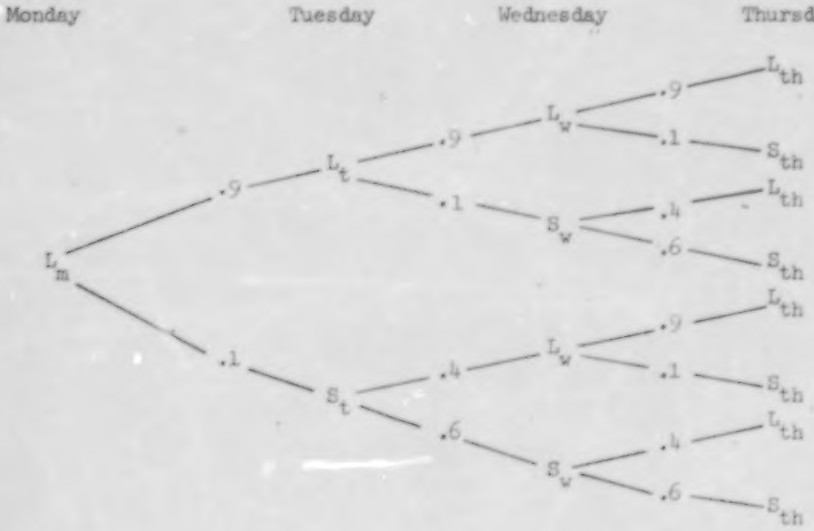
6. Thus, if she cooks L for 1000 days, she cooks S, during that time, on approximately ______ days.

$\frac{1000}{1250}$; .8

7. Hence, P(cooking L) is about $\frac{1000}{\boxed{}}$, or ______.

Again, we can use a tree to compute probabilities in this situation.

8. Draw an appropriate tree, and find the probabilities for Lumpies for Monday, Tuesday, Wednesday. Compare your results with those given below.


Monday
Tuesday
Wednesday
Thursday
L_m
.9
.1
L_t
S_t
.9
.1
.4
.6
L_w
S_w
L_w
S_w
.9
.1
.4
.6
.9
.1
.4
.6
L_{th}
S_{th}
L_{th}
S_{th}
L_{th}
S_{th}
L_{th}
S_{th}

$P(L_m) = 1$

$P(L_t) = .9$

$P(S_t) = .1$

$P(L_w) = .9(.9) + .4(.1) = .85$

$P(S_w) = 1 - .85 = .15$

.4

9. $P(L_{th}) = .9\ P(L_w) +$ ______ $P(S_w)$.

This is true because she cooks L on Thursday if either:

she cooks L on Wednesday and does not burn it; or
she cooks S on Wednesday and burns it.

.825

10. We may conclude:

$P(L_{th}) = .9(.85) + .4(.15) =$ ______.

Exercise. (Answers on page 347.)

1. Find the probability that she cooks L on Friday.

Now let us examine our results. We have found that the probabilities of cooking L are:

Day	Probability she cooks L
Monday	1.
Tuesday	.9
Wednesday	.85
Thursday	.825
Friday	.8125

less

As we see, the probability of cooking L changes from day to day.

11. In fact, each day the probability of cooking L is ______ (less, more) than that of the previous day.

12. On the other hand, the probabilities do not decrease very rapidly. That is, if we subtract each day's probability from the previous one, we see the differences shown below:

	Day	Probability she cooks L	Decrease in probability from previous day
	Mon.	1.	
.1	Tues.	.9	1 - .9 = ______.
.05	Wed.	.85	.9 - .85 = ______.
.025	Thurs.	.825	.85 - .825 = ______.
.0125	Fri.	.8125	.825 - .8125 = ______.

Look carefully at the numbers you found in the last box:

.1
.05
.025
.0125

Do you see a pattern? Try to find it before going on.

.5 (or $\frac{1}{2}$)	13. .1 × ______ = .05.
.5	14. .05 × ______ = .025.
.5	15. .025 × ______ = .0125.
.0125 × .5, .00625	16. We might make a guess: The number after .0125 is .0125 × ______ = ______.

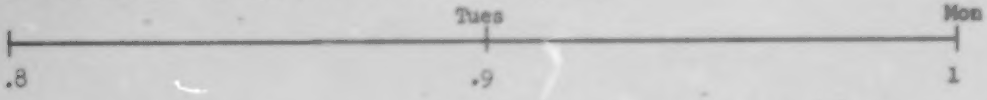
You can probably guess, therefore, that we could extend our list of probabilities of L.

Your guess: From Friday to Saturday the probability of cooking L decreases by .00625.

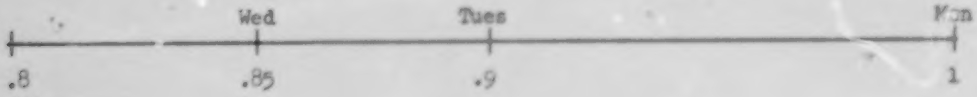
.80625

17. Hence, you guess: The probability of cooking L on Saturday is ______.

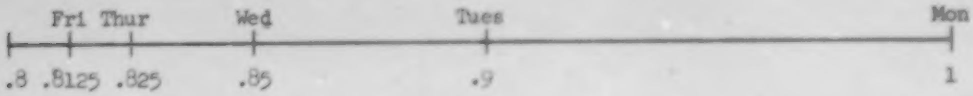
Let us make a number line diagram of the probabilities of cooking L. We will use only the interval between .8 and 1, since this is the interval where all our probabilities lie. We first locate and label the probabilities of "cooking L on Monday" and "cooking L on Tuesday".



To locate Wednesday's probability, we must move to the left. Take $\frac{1}{2}$ (that is, .5) of the interval from .9 to 1 and move this distance to the left.



Repeating this process, we see:



Think about what would happen if you computed several more days' probabilities of cooking Lumpies.

For each question, state what you think would happen.

decrease

18. Even if you went on for a month, the probability of cooking Lumpies would continue to ______________ (increase, decrease).

would not

19. You ______________ (would, would not) reach a day, however, when the probability of cooking Lumpies was less than .8.

Think of what you would expect to see if you showed, on the number line, the probability of cooking Soggies on various days.

1

20. For any day, the probability of cooking S is found by subtracting the probability of cooking L from ______.

increase

21. As the days wear on, the probabilities of cooking Soggies ______________ .
(increase,decrease)

.2

22. They always remain less than ______, but they are probabilities very, very close to this number.

We have only guessed at the results summarized above. In a later section we will find out a little more about how we might have proved that they are true.

Exercises. (Answers on page 347.)

2. After a month, what is the probability that the family gets burned cereal for breakfast? (Assume in this and the following exercise that the probability of L for breakfast is .8. It is, in fact, very close to .8.)

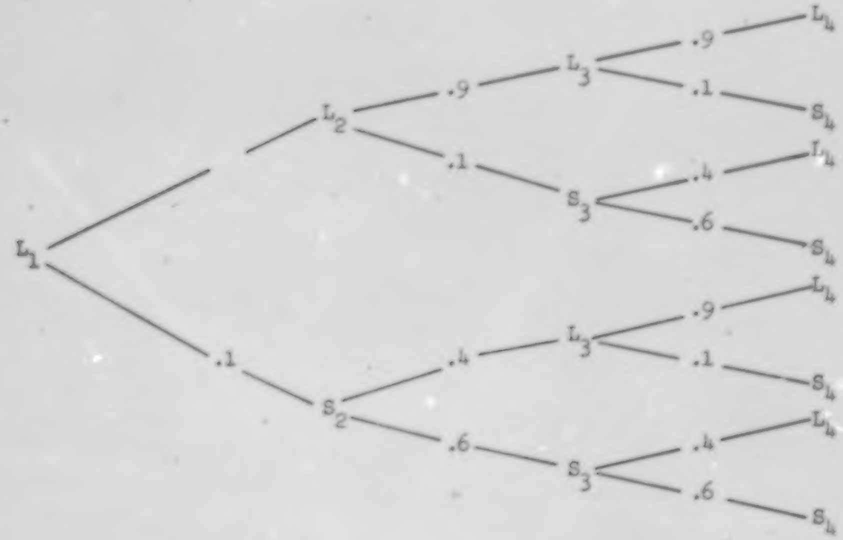
3. The family wakes up one morning to find that the cereal is burned. What is the probability that it was Lumpies?

4. We supposed that this cook began by cooking Lumpies. Thus, her probability of cooking L on the first day (Monday) was 1. Suppose instead that the cook begins by cooking Soggies on the first day. Thus, the probability that she cooks L on Monday is 0. Think about what the probability is of cooking L on Tuesday, Wednesday, etc. Try to guess what sort of numbers you would find for these probabilities. (You may wish to compute them.) Estimate the probability of cooking L on Saturday.

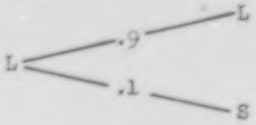
12-4. Markov Chains

Let us examine once more the tree we drew in Section 12-3. Let us recall also our work in the problems in that section.

We will copy the tree, with one slight change. We will write L_1 (Lumpies on first day), rather than L_m (Lumpies on Monday). Likewise, we will write S_2 instead of S_t, etc. Our tree becomes:



1. Note that from each L we have two paths



associated with the conditional probabilities

.1 .9 and ______.

2. This follows from the fact that

.9 P(L one day|L preceding day) = ______,

while

.1 P(S one day|L preceding day) = ______.

3. Likewise, from each S we have two paths, associated

.4, .6 with the conditional probabilities ______ and ______.

Indeed, a tree for this kind of situation, showing as many days as we wish, can be drawn step by step as soon as we have the answers to two questions:

1. Does the process begin with S or with L ?
2. Given what a particular letter is (for example, given that the third letter is S), what is the probability that the next letter is S ? (If we know this, of course, we can also find the probability that the next letter is L. We need only to subtract P(S) from 1.)

Problems of this sort occur very often. Consequently, mathematicians have developed some special definitions to use in describing them.

In such situations we have a sequence of trials. The result of each trial is called a state. (In our example, states at each trial are S and L.) The initial state is known. After the first state the probability of a specified state for a particular trial depends only on what happened the time just before. That is, we know the probability of going from each state to each state. These probabilities are called transition probabilities. A process that has these properties is called a Markov Process.

We can list the transition probabilities for the tree in a table.

If process is in state	Probability of going next to state	
	L	S
L	.9	.1
S	.4	.6

.6 — 4. The probability of going from state S to state S is ______.

This is P(not burning S|she cooks S).

1 — 5. In each row the sum of the entries is ______.

An array of numbers, such as

$$\begin{pmatrix} .9 & .1 \\ .4 & .6 \end{pmatrix}$$

is often called a matrix. (A matrix, in general, is a rectangular array of numbers.)

In Section 12-1, we had to do with a cook who burns Lumpies with probability .1. Each time she burns L, she cooks S. After cooking S once, however, she always returns to L.

Again we can regard "cooking L" and "cooking S" as the two possible states.

6. As in the previous example, make a table showing the transition probabilities. Check your result with the completed table below.

If process is in state	Probability of going next to state	
	L	S
L	.9	.1
S	1	0

0 7. The probability of going from state S to state S is ______.

This is because after cooking S she always returns to L.

1 8. In each row the sum of the entries is ______.

Exercises. (Answers on page 349.)

1. A cook always burns the Lumpies and the Soggies. Each time she burns one, she changes to the other. Give the matrix of transition probabilities.
2. A cook changes cereal whenever she burns it. If her transition probabilities are $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and she begins with L, what happens? What happens if she begins with S ?

12-5. Experiment--Mixtures (Discussion on page 311.)

Put two black marbles in an urn labeled X and two white marbles in an urn labeled Y. Our process is as follows: Draw a marble from each urn. Put the one from X into Y, and put the one from Y into X. (Now, of course, you have a black and a white marble in each urn.) Repeat, each time recording the number of black marbles in urn X.

1. Continue until you have a sequence with 30 entries.
2. Can you find in your sequence a 2 followed at once by 0, or 0 followed at once by 2 ? Why not?
3. If there is exactly 1 black marble in urn X, then how many white marbles are there in X ? How many black marbles and how many white marbles are there in Y ?

We found that we had 1's more frequently than 0's or 2's. We were not surprised at all. We are sure that you, too, found this result.

1 1. Each 2 is followed by a ______.

1 Each 0 is followed by a ______. (Except, of course, for the last number of your sequence.)

2. Suppose you repeated this process many, many times without counting how many. Suppose your last number was 1. Suppose, then, that you found you had twenty 0's and sixteen 2's. You could be certain that your

36 number of 1's was greater than ______. (Give the best estimate you can make.)

Now you should see (if you didn't before) how we could be so sure that you had 1's more frequently than 0's or 2's. In fact, it is highly probable that you had 1's more frequently than both together.

3. Would the results

21012 12101 21210 12101
01012 10121

yes be possible for this experiment? ______

no 4. Would they be very likely? ______

Let us be sure we understand why.

A "1" means: We have 1 white and 1 black marble in X, and 1 white and 1 black marble in Y.

$\frac{1}{2}$ 5. When we have this situation, the probability of getting 1 again is ______. (We can draw a white from X and a white from Y, or a black from X and a black from Y.)

14 6. The result shown in Item 3 is very unlikely. In it, 1 is followed 14 times by something other than 1, and never by 1. This result is as unlikely as throwing heads _____ times in a row with a coin. (See item 5.) Which doesn't happen often!

Exercises. (Answers on page 349.)

1. We can think of the number of black marbles in urn X after each step as describing the state of the process after this step. How many states are there?
2. Draw a tree showing 4 trials.
3. Construct a matrix of transition probabilities for this process.
4. What is the probability of having 1 as the first state? as the second? as the third? as the fourth?

It is easy to compute the probabilities for 1 at various steps. The process becomes clearer if we use symbols fitted to our purpose. Let us write p_1 for P(1 as first state), p_2 for P(1 as second state).

third 7. p_3 = P(1 as ______ state).

The 3 in p_3 is called a subscript, because it is written below.

p_4	8. For the probability that the fourth state is 1, we naturally write ______.
	9. In exercise 4 you found:
0	p_1 = ______ ;
1	p_2 = ______ ;
$\frac{1}{2}$	p_3 = ______ ;
$\frac{3}{4}$	p_4 = ______ .
$1 - p_4$, or $\frac{1}{4}$	10. The probability that the fourth state is not 1 is 1 - ______.

We know that

$$\begin{aligned} p_5 &= P(\text{fifth state is } 1) \\ &= P(\text{fourth state is } 1 \cap \text{fifth state is } 1) + P(\text{fourth state is not } 1) \\ &= \frac{1}{2}p_4 + (1 - p_4) \\ &= 1 - \frac{1}{2}p_4 . \end{aligned}$$

The same reasoning can be used for any state. If p_n is the probability the nth state is 1, then p_{n+1} is the probability the next state is 1, and

$$p_{n+1} = 1 - \frac{1}{2}p_n .$$

We have found a formula which enables us to go from the probability of 1 at the nth step to the probability of 1 at the next--the (n+1)st--step. Such a formula, which enables us to find a result for n+1 if we know the result for n, is called a recursion formula. With it, along with our knowledge of p_1, we are able to find the probability of 1 at various states. (You may recall the way in which we built up the Pascal triangle line by line. Such a step-by-step process is called a recursive process.)

Exercise. (Answer on page 349.)

5. Find p_7 .

Suppose we wished to find p_{1000}. We could do this, of course, with the recursion formula, finding first p_8, then p_9, etc. (We have already found p_7 in Exercise 5.) There are two other ways, however, by which we might proceed. See whether you can find them; then go on.

You might have looked for a pattern in the probabilities we have. Examine the following table.

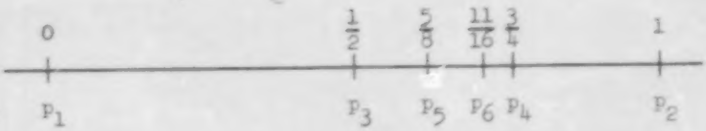
p_1	0
p_2	1
p_3	$\frac{1}{2}$
p_4	$\frac{3}{4}$
p_5	$\frac{5}{8}$
p_6	$\frac{11}{16}$
p_7	$\frac{21}{32}$

Perhaps you added to this table p_8, p_9, p_{10}. By examining the values of the p's, you could have seen that all are close to $\frac{2}{3}$. In fact, we have:

p_1	$0 = 0$	$\left\lvert p_1 - \frac{2}{3}\right\rvert = \frac{2}{3}$
p_2	$1 = 1.0$	$\left\lvert p_2 - \frac{2}{3}\right\rvert = \frac{1}{3}$
p_3	$\frac{1}{2} = .5$	$\left\lvert p_3 - \frac{2}{3}\right\rvert = \frac{1}{6}$
p_4	$\frac{3}{4} = .75$	$\left\lvert p_4 - \frac{2}{3}\right\rvert = \frac{1}{12}$
p_5	$\frac{5}{8} = .625$	$\left\lvert p_5 - \frac{2}{3}\right\rvert = \frac{1}{24}$
p_6	$\frac{11}{16} = .6875$	$\left\lvert p_6 - \frac{2}{3}\right\rvert = \frac{1}{48}$
p_7	$\frac{21}{32} = .65625$	$\left\lvert p_7 - \frac{2}{3}\right\rvert = \frac{1}{96}$
p_8	$\frac{43}{64} = .671875$	$\left\lvert p_8 - \frac{2}{3}\right\rvert = \frac{1}{192}$
p_9	$\frac{85}{128} = .6640625$	$\left\lvert p_9 - \frac{2}{3}\right\rvert = \frac{1}{384}$
p_{10}	$\frac{171}{256} = .66796875$	$\left\lvert p_{10} - \frac{2}{3}\right\rvert = \frac{1}{768}$

Some or all of this data may have led you to conclude: The p's are all different. Howeve , we would expect p_{1000} to be close to $\frac{2}{3}$.

Note the number line representation.



11. On the number line we see

$$p_1 < \_\_\_\_\_ < p_5 .$$

p_3

12. $p_2 \underset{(<,\ >)}{\_\_\_\_\_} p_4 \underset{(<,\ >)}{\_\_\_\_\_} p_6$.

$p_2 > p_4 > p_6$

13. p_5 ______ p_6 .

$p_5 < p_6$

Looking at our number line, we would expect:

$$p_5 < p_7 < p_8 < p_6 .$$

If you check, you will see that this is indeed true.

You might also have reasoned in quite another way. You could have thought: Suppose we repeat this process many times.

14. If we got 2's a hundred times, we'd also expect ____'s about 100 times.

0's

15. We'd expect to change from 1 to 2, then, about 100 times. We'd also expect to change from 1 to 0 about ______ times.

100

16. So there would be about ______ times when 1 was followed by something different from 1.

200

17. But we have noted that if we get 1, then the probabilitity of getting 1 the next time is ______.

$\frac{1}{2}$

$\frac{1}{2}$	18. Likewise, when we get 1, then the probability of not getting 1 next time is ______.
400	19. Combining this fact with the conclusion in Item 16, we would expect 1 about ______ times in all.

We may conclude: If we get 2's a hundred times, we would expect 0's about 100 times and 1's about 400 times. Eventually the probabilities are about:

$\frac{1}{6}$	20. $P(2) = P(0) =$ ______.
$\frac{4}{6}$ or $\frac{2}{3}$	21. $P(1) =$ ______.

This does not mean that in our experiment the probability is exactly $\frac{2}{3}$ that the tenth number, for example, is 1. However, though the probabilities of having 1 differ from one entry to the next, they behave in a way that makes our estimates reasonable.

12-6. The King's Choice.

Of the subjects of a certain king, not all are truthful. In fact, if a subject is selected at random, the probability that he always tells the truth is $\frac{2}{3}$. However, the probability that he never tells the truth is $\frac{1}{3}$. The king of this country is trying to decide whom to marry. There are only two possible choices, Princess Anne and Princess Barbara. One day the king whispers to one of his subjects that his choice is Anne. This confidant hastily whispers to another person, "The king has chosen ______." Which name he says depends, of course, on whether or not he is truthful. So it goes. Each person, when he hears the rumor, whispers either the name he hears or the other to someone who has not heard. Eventually, 12 people have heard the rumor. But what has the 12th one heard? Has he heard the truth or not?

Probably, you would say that you can't be sure. But, then, of course, you might find the probability that the 12th person has heard the rumor. What do you think it is? Record your guess.

Experiment. (Discussion on page 312.)

If there are as many as 12 students in your class, how can you do as a group an experiment which would duplicate the history of this rumor?

Here is one possible plane. Twelve students are selected. The teacher says, "A" to the first student; this student throws a die. If it shows 1, 2, 3 or 4, he says to the next student the letter he heard. (He is truthful.) If it comes up 5 or 6, he says the letter "B". (He is not truthful.) Each student repeats the same process until the 12th is reached.

You can use, in place of a die, a spinner, or an urn with 2 black marbles and 1 white marble.

You can try the experiment by yourself. Here is a record of 12 trials, using the die-throwing plan described above. (A throw of 1, 2, 3, or 4 means "no change in letter". A throw of 5 or 6 means "change letter".)

Letters	A	↑ B	↑ A	↑ B	B	B	B	B	B	↑ A	A	↑ B
Die Result		6	5	6	2	2	3	1	4	6	3	6

The arrows are inserted to show that throws of 5 or 6 signal changes in letter.

1. Try for yourself. Record at least 3 results--that is, at least 3 sequences of 12 letters.
2. In each case, your row of letters begins with A. Why?
3. Out of several runs of this experiment, in about what part of them would you expect the second letter to be A ?
4. What is the matrix of transition probabilities?
5. Incidentally, suppose the king is untruthful. Is there anything that makes you able to decide whether he is or not? Does it matter in the problem?

6. After 100 repetitions of this experiment, finding 12 letters in each, approximately how many times would you expect to find A as the twelfth letter?

7. Suppose just one of the first 11 people is untruthful. Then what is the 12th letter? What is it if just two people are untruthful?

Let us find the probability that the 12th person hears A. Our previous results suggest that we might look for a recursion formula.

Let us use p_1 for P(first letter is A), p_2 for P(second letter is A), etc.

1. The king said his choice was Anne. Hence,

$$p_1 = P(\text{first letter is A}) = \underline{\qquad}.$$

1

2. The second letter is A if the first person is truthful. Hence,

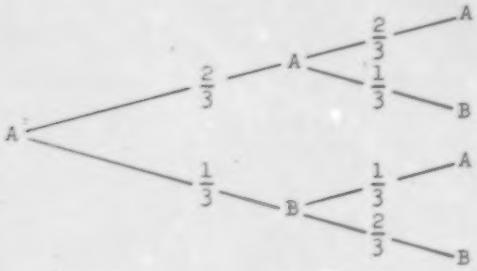
$$p_2 = \underline{\qquad}.$$

$\frac{2}{3}$

3. The probability that the second letter is B is $\underline{1 - \qquad}$.

$1 - p_2$, or $1 - \frac{2}{3}$

4. Draw the tree which will help you find p_3. Check with the one shown below.



$\frac{1}{3}$

5. $p_3 = \frac{2}{3} p_2 +$ ______ $(1 - p_2)$.

6. This is true because the 3rd letter is A if either the second letter is A and the second person is truthful,

or the second letter is B and the second person is ______.

untruthful

7. The same reasoning used in Item 6 shows that

$$p_4 = \frac{2}{3} p_3 + \text{\_\_\_\_\_\_}.$$

$\frac{1}{3}(1 - p_3)$

8. Again, we have a ______ formula.

$$p_{n+1} = \frac{2}{3} p_n + \frac{1}{3}(1 - p_n).$$

recursion

Applying the distributive property in Item 8, we have:

$$p_{n+1} = \frac{2}{3} p_n + \frac{1}{3} - \frac{1}{3} p_n$$

$$p_{n+1} = \frac{1}{3}(p_n + 1).$$

Exercises. (Answers on page 350.)

1. Find $p_3, p_4, p_5, p_6, p_7, p_8$.

2. What is the approximate value of p_{12} ?

The method outlined above is not the only one you might have used. You might have reasoned: Each time an untruthful person hears a letter, he changes it. But when a truthful person hears a letter he repeats it.

9. If the number of changes is odd, then the last letter is ______.

B

10. But if the number of changes is ______, the last letter is A.

even

Hence, p_{12}, the probability that the last letter is A, is the probability that an even number of people changed the rumor. Hence, to find p_{12} we could add the probability that of the 11 people who passed on the rumor, all 11 were truthful; the probability that 9 of the 11 were truthful, etc.

If you have completed Chapter 9, you should recognize, then, that

$$p_{12} = (\tfrac{2}{3})^{11} + \binom{11}{9}(\tfrac{2}{3})^{9}(\tfrac{1}{3})^{2} + \binom{11}{7}(\tfrac{2}{3})^{7}(\tfrac{1}{3})^{4} + \binom{11}{5}(\tfrac{2}{3})^{5}(\tfrac{1}{3})^{6} + \binom{11}{3}(\tfrac{2}{3})^{3}(\tfrac{1}{3})^{8} + \binom{11}{1}(\tfrac{2}{3})(\tfrac{1}{3})^{10} .$$

This would be tedious to compute directly, but if you had adequate tables it would be easy to find p_{12}.

Again you would find: After that rumor has been whispered from courtier to knight to square to page to cook to beggar to soldier to sailor to tinker to tailor, the probability that it is the truth is approximately $\frac{1}{2}$. And that's not all. The same procedure could be followed for any ratio of truth-tellers to liars (provided there is at least one liar), and for larger numbers of people. In the long run, the chance that the person will hear the truth is $\frac{1}{2}$. Just remember, the next time you hear some wild rumor, that the truth may be harder to come by than you think.

APPENDIX

THE LAW OF LARGE NUMBERS

The study of probability begins with simple intuitive ideas. We feel intuitively, for example, that out of a large number of tosses of a coin, we are "almost certain" to get heads "about half the time". Our study of probability has prepared us to explain more fully this intuitive idea. In particular, we can give more precise meaning to the phrases "large number of tosses", "almost certain","about half the time".

Tossing a coin many times can be regarded as a series of Bernoulli trials. If we regard "heads" as "successes", the probability p of success on a single trial is $.5$.

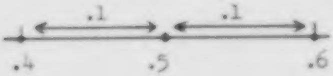
Let us think, for example, of 10 trials. We know how to compute the probability distribution for Bernoulli trials for $n = 10$, $p = q = .5$. We know:

(1) The most likely outcome is 5 successes.

(2) However, the probability of exactly 5 successes is not great. (It is, in fact, .246.)

(3) On the other hand, we are quite likely to get 4, 5, or 6 successes. (The probability is .656.)

We can state (3) in terms of the average number of successes per trial in 10 trials -- which we have called m. From (3), the probability is .656 that m is either .4, .5, or .6; that is, that m differs from .5 by at most .1.



We may write all this more concisely:

$$P(\,|m - .5| \leq .1\,) = .656$$

Notice that in the preceding line we used the symbol $|m - .5|$, which is read: "absolute value of $m - .5$". It is the *distance* between m and $.5$. If $|m - .5| \leq .1$, then m lies in the number line region shown above. For 10 trials, the probability that m lies in this region is .656.

Now, suppose that we consider 100 trials. Though you know how to find the probability distribution, the lengthy computations that you would need are dismaying. One question we have left for later courses is: How can we *efficiently* find probabilities of this kind?

We can tell you, however, some results. (Compare them with (1) to (3) above.) If we toss a coin 100 times:

(1a) The most likely outcome is 50 heads.

(2a) However, the probability of getting exactly 50 head is small -- smaller than the probability of 5 heads in 10 trials. (Recall the "flattening" of the binomial distributions with increasing n.)

(3a) *Moreover*, the probability of getting 49, 50, or 51 heads is not very great. It is not as great as that of getting 4, 5, or 6 heads in 10 trials.

But, we can add something new. The probability that the *average* number of heads per throw is near .5 is *greater* for 100 throws than for 10. For 100 throws,

$$P(\,|m - .5| \leq .1\,) \sim .95 .$$

That is, for 10 throws it is fairly likely (probability is .656) that the number of heads in 4, 5, or 6. For 100 throws, it is more likely (probability .95) that the number of heads is between 40 and 60.

We could go on, considering 1000 throws, 10,000 throws, etc. By now, you should anticipate the next reasonable step -- we look for a trend.

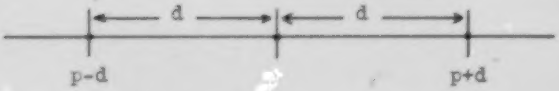
In connection with Markov chains, we have had some experience in examining trends in a set of probabilities. There, we encountered some easily noted number patterns in a sequence of probabilities. In our present situation, we do not have such engaging patterns.

We have, however, a general tendency which is clear. When the number of throws is very large, then $P(|m - .5| \leq .1)$ is very close to 1.

Let us be more specific about what this means. Suppose you select a positive number less than 1. You might choose, for example, .99. Then we can find a number n with the following property. If the number of trials is n or more, $P(|m - .5| \leq .1) > .99$. If you had chosen .9999, we would not have been daunted. Again we find how many trials you would need to insure that

$$P(|m - .5| \leq .1) > .9999 .$$

Of course, you will realize that our discussion could have been carried out with any specific value of p. It was not necessary to use $p = .5$. Nor was it necessary to use .1. In fact, our result is very general. Let d be any appropriate positive number. Consider the region:



No matter how many trials you decide to use, you cannot be certain that m will lie in the region. But, you can be almost certain if you choose a large enough number of trials.

This fact is sometimes called The Law of Large Numbers.

Remarks:

(1) Notice that the Law of Large Numbers has to do with the average number of successes, not simply the number of successes.

(2) Note also that the Law of Large Numbers does not tell us that something is certain. It is stated in terms of probabilities. It says: For a very large number of trials, it is extremely likely that the average number of successes is close to p.

Probability

The Law of Large Numbers is widely applicable. Suppose we have an experiment for which the probability of some event E is $\frac{2}{5}$. We can consider the long sequence of independent trials of this experiment. For each trial, we record "success" when event E occurs, "failure" when it does not. The probability of "success" is $\frac{2}{5}$.

We say "if the experiment is repeated a large number of times, then E is almost certain to occur about $\frac{2}{5}$ of the time." More precisely, the Law of Large Numbers implies that:

(1) If we decide how small we want $| m - \frac{2}{5} |$ to be (that is, if we choose some d, with $d > 0$);

(2) if we decide how nearly certain we want to be (that is, how close to 1 we want $P (| m - \frac{2}{5} | \leq d)$ to be); then

(3) we can find how many trials are required to fix our requirements.

You may ask: "How do we find out how many trials are needed ?" The Law of Large Numbers only tells us that the necessary number of trials can be found. How to do so is another matter that we defer to later courses.

Reasoning similar to that used above can be applied to situations in which we do not know the value of P. In such cases we wish to estimate P to a degree of accuracy appropriate to the application. Once again, we can do so. By using a large enough number of trials, we can be almost certain that our estimate will be accurate.

The fact that a suitable number of trials can be found makes probability an essential tool of the opinion sampler, the scientific experimenter, the industrial engineer charged with maintaining efficient production.

Uncertainty cannot be translated into certainty by probabilistic techniques. But probabilistic techniques do enable us to decide, in situations involving uncertainty, that some events are overwhelmingly likely.

Discussion of Experiment, Section 8-1

Our results:

	Number Recorded		Number Recorded
I	16	R	10
		B	6
II	14	R	7
		B	7

We expected about an equal number of I's and II's. We expected about $\frac{2}{3}$ of the I's and $\frac{1}{2}$ of the II's would be R. As you see, our results agree rather closely with our expectations.

Notice that we obtained a total of 17 R's on the 30 trials. As you will see (Items 8 to 14), for this experiment $P(R) = \frac{7}{12} \sim .58$. Our result, $\frac{17}{30} \sim .53$, is remarkably close. Your results may differ from ours, of course.

Discussion of Experiment, Section 9-1

Of course, as usual, your experiment is almost certainly not like the authors'. Here are our results:

First Throw	Second Throw	Third Throw	Score
6	3	5	2
2	4	1	0
2	5	2	1
2	4	3	0
6	4	1	1
2	5	3	1
1	3	4	0
4	5	2	1
3	2	2	0
5	1	6	2

1. For our results, we observe:

Score	0	1	2	3
Number of times	4	4	2	0

Were yours similar?

2. We got mostly 0's and 1's, and we weren't surprised. You win only if you throw a 5 or 6. On each throw you're less likely to get 1 than you are to get 0. We expect a low score to occur more often than a high score.

3. Our average was .8, again indicating that low scores are more frequent than high ones. We found this average by dividing the total score, 8, by the number of games, 10. This is not the same as the average of 0, 1, 2, and 3. Naturally not--we get some scores more often than others.

4. When you play the game with the die your probability of winning a point on a throw is $\frac{1}{3}$, and of not winning is $\frac{2}{3}$. If the spinner is such that the red area is $\frac{1}{3}$ the total area, then red and blue have probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Hence, we can use the spinner for the game, scoring 1 point when the spinner stops on red. It is only the probabilities $\frac{1}{3}$ and $\frac{2}{3}$ that matter, not the fact that a die is used.

Discussion of Experiment, Section 10-1

Our results for 10 trials were:

Trial \ Match	A	2	3	4	5	6	Total
1	1		1			1	3
2							0
3						1	1
4			1				1
5							0
6					1		1
7		1		1			2
8							0
9	1						1
10							0

We scored 9 points in 10 games. On this evidence, we might expect to score approximately 90 points in 100 games. However, we recognize that 10 trials aren't very many, so we would not have much confidence in our guess.

Discussion of Experiment, Section 11-1

Our record was:

A B B B B B B A B A B B A A B A A A B A A A A A A

When we begin counting, we get 1, 0 -- and this is enough to tell us that A did not always lead on our vote. In fact, after the second vote the vote was tied.

1. If the first vote is B, then A is behind on the first vote. A leads on the first vote only if the first vote is A. If the first two votes are A B, then A and B are tied on the second vote, and A does not lead. A leads on the first two votes only if the first two votes are A A.

2. $P(A\ A) = \frac{14}{25} \cdot \frac{13}{24} \approx .3$. (This is exactly like an urn problem in drawing without replacement.)

3. If the first two votes are A A, then A leads by 2 votes. He is certain to be still ahead after the first 3 votes, so $P(\text{A ahead after 3 votes}) \approx .3$.

4. Your answer to Exercise 2 leads you to conclude that the probability that Arthur does not lead after two votes is .7 . Thus it seems unlikely that he will lead throughout the ballot count.

Discussion of Experiment, Section 12-1

1. Here is how our record looked:

Day	Cereal cooked	Burned Yes/No
Mon., Jan. 1	L	No
Tues.	L	No
Wed.	L	No
Thurs.	L	No
Fri.	L	Yes (Here we draw a red marble.)
Sat.	S	No (Automatically go back to L.)
Sun.	L	No
Mon., Jan. 8	L	No

You may go on a long time before you get an S. This is the sort of situation where a great number of trials would be necessary to give you results enough to base any conclusions on. If you do the experiment a few times, however, it will help you to understand the situation better. This was our main purpose in asking you to do it.

Discussion of Experiment, Section 12-3

1. Use, as before, two jars, one labeled L and one S. Place 9 white marbles and one red marble in the L jar. Place 6 white and 4 red marbles in the S jar. Draw a marble from L. If it is white, put it back and draw another marble from L. If it is red, put it back and draw a marble from S. Continue. After each draw, put the marble back in the *same* jar. Whenever the marble you draw is *white*, draw the next marble from the *same* jar. (That is, cook the *same* cereal.) Whenever the marble is red, draw the next marble from the *other* jar. (You may have worded your answer differently, but the ideas should be the same.)

   Our record of what cereal she cooks for 30 days:

   LLLLL LSLLL LLLLL LLLLL LLLSS SLLSS

2. (a) She certainly cooks L on Monday (probability is 1). The probability that she cooks L on Tuesday is .9. So, clearly, the probability of cooking L is not the same *each* day. We would expect from our results in the last section that the probability of cooking L differs from one day to the next.

   (b) In your experiment, you should observe that you get, as a rule, several L's in a row, then some S's, then more L's, etc. If you repeat this experiment many times, you find that the rows of L's tend to be longer, on the average, than the rows of S's. This is not surprising. If she cooks L, she is less likely to burn it than if she cooks S.

3. Each time she burns one kind, she changes to the other. So each change in cereal corresponds to one burning. She starts with L, then changes to S, then back to L, etc. There are, consequently, approximately as many burnings of L as of S.

This last result might surprise you at first. She is more likely to burn S (if she cooks it) than to burn L (if she cooks it). However, she cooks Lumpies more often. Consequently, the family gets about as much burned Lumpies as burned Soggies.

Discussion of Experiment, Section 12-5

1. You will have a sequence made up of the numbers 0, 1, 2. The first two terms are 2, 1 and the third may be 2, 1, or 0. Here are our results:

   21101   21110   11012   11110   11010   21211

2. No. You can change the number of black marbles by at most 1 in a single step.

3. 1 white in X; 1 black and 1 white in Y.

Discussion of Experiment, Section 12-6

1. Here are our results:

(1) Letters A ↑ B ↑ A ↑ B B B B B B ↑ A A ↑ B

Die Result 6 5 6 2 2 3 1 4 6 3 6

(2) Letters A A ↑ B B B B B B ↑ A ↑ B ↑ A A

Die Result 1 5 3 2 1 2 4 5 6 6 2

(3) Letters A A A A ↑ B B ↑ A ↑ B B B B B

Die Result 4 2 3 6 2 6 5 3 1 2 4

2. We are always supposing that the king whispers to the first person, "My choice is Anne." Hence, the first person heard A.

3. Since the probability that a person is truthful is $\frac{2}{3}$, you would expect the second letter to be A about $\frac{2}{3}$ of the time. Is this the case for your experiments?

4.

$$\begin{array}{c|cc} & A & B \\ \hline A & \frac{2}{3} & \frac{1}{3} \\ B & \frac{1}{3} & \frac{2}{3} \end{array}$$

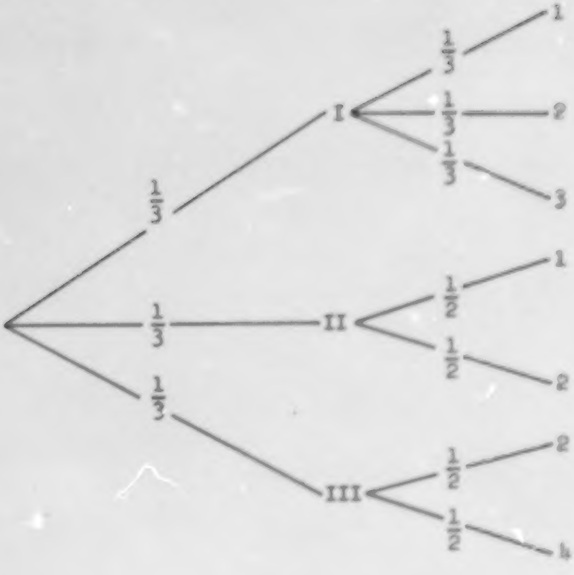
5. All we really know is what the king tells the first person. If the king is untruthful, then, of course, he says "Anne" when his choice is really Barbara. But this will not have any effect on what happens later to the rumor.

6. Watch for a catch! You might think that because more people are truthful than untruthful, A would be more likely (since A is really what the king said). But look at the next question.

7. If just one of the 11 people who pass the rumor on is untruthful, then you can be sure the last person hears B. One person says the wrong name, and the rest truthfully pass on what they heard. But if just 2 people in the chain are untruthful, then one changes the name, and the other changes it back. It appears that whether the twelfth person hears A or B depends on whether an odd or an even number of truthful people passed the rumor on. Does this change your opinion about what the answer to Exercise 6 should be? You will find out the correct answer to Exercise 6 as you read on in this section.

Answers to Exercises 8-2

1.



I
II
III
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
$\frac{1}{2}$
1
2
3
1
2
2
4

$P(I \cap 1) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(I \cap 2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(I \cap 3) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(II \cap 1) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(II \cap 2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(III \cap 2) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(III \cap 4) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(1) = P(I \cap 1) + P(II \cap 1) = \frac{1}{9} + \frac{1}{6} = \frac{5}{18}$

$P(2) = P(I \cap 2) + P(II \cap 2) + P(III \cap 2) = \frac{1}{9} + \frac{1}{6} + \frac{1}{6} = \frac{8}{18} = \frac{4}{9}$

$P(3) = P(I \cap 3) = \frac{1}{9}$

$P(4) = P(III \cap 4) = \frac{1}{6}$

Check: $\frac{5}{18} + \frac{4}{9} + \frac{1}{9} + \frac{1}{6} = \frac{5}{18} + \frac{8}{18} + \frac{2}{18} + \frac{3}{18} = \frac{18}{18} = 1.$

2. $P(\text{even}) = P(2) + P(4) = \frac{4}{9} + \frac{1}{6} = \frac{11}{18}$

$P(\text{less than } 3) = P(1) + P(2) = \frac{5}{18} + \frac{4}{9} = \frac{13}{18}$

$P(\text{even or less than } 3) = P(\text{even}) + P(\text{less than } 3) - P(\text{even } \underline{\text{and}} \text{ less than } 3)$

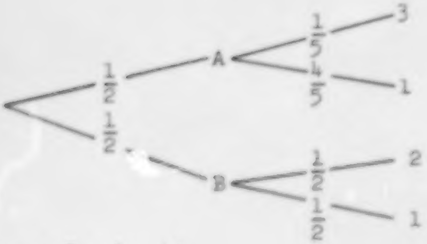
$= \frac{11}{18} + \frac{13}{18} - \frac{4}{9} = \frac{16}{18} = \frac{8}{9}$

Note: (1) $P(\text{even and less than } 3) = P(2) = \frac{4}{9}$

(2) We also observe

$P(\text{even or less than } 3) = 1 - P(3) = 1 - \frac{1}{9} = \frac{8}{9}$.

3.



$P(A \cap 3) = \frac{1}{10}$

$P(A \cap 1) = \frac{2}{5}$

$P(B \cap 2) = \frac{1}{4}$

$P(B \cap 1) = \frac{1}{4}$

$P(1) = \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$

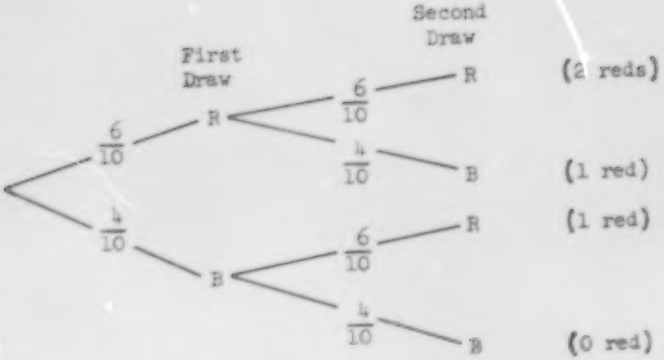
$P(2) = \frac{1}{4}$

$P(3) = \frac{1}{10}$

Check: $\frac{13}{20} + \frac{1}{4} + \frac{1}{10} = \frac{13}{20} + \frac{5}{20} + \frac{2}{20} = 1$.

Note: If the boy plays this way 20 times he will (on the "average") receive 1 piece 13 times, 2 pieces 5 times and 3 pieces twice. This is a total of 13 + 10 + 6 = 29 pieces. In some sense we might say for a given play he should "expect" to receive $\frac{29}{20} = 1.45$ pieces. (See Chapter 10.)

4.


Second Draw
First Draw
$\frac{6}{10}$
R
$\frac{6}{10}$
R
(2 reds)
$\frac{4}{10}$
B
(1 red)
$\frac{4}{10}$
B
$\frac{6}{10}$
R
(1 red)
$\frac{4}{10}$
B
(0 red)

$P(0) = \frac{4}{10} \cdot \frac{4}{10} = \frac{16}{100} = \frac{4}{25}$

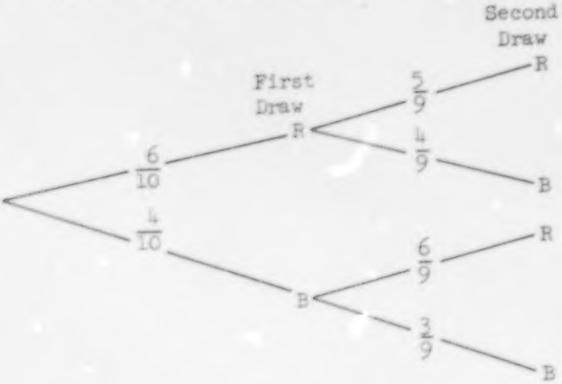
$P(1) = \frac{6}{10} \cdot \frac{4}{10} + \frac{4}{10} \cdot \frac{6}{10} = \frac{48}{100} = \frac{12}{25}$

$P(2) = \frac{6}{10} \cdot \frac{6}{10} = \frac{36}{100} = \frac{9}{25}$

Check: $P(0) + P(1) + P(2) = \frac{4}{25} + \frac{12}{25} + \frac{9}{25} = 1$

Note: This example is one of a general class of problems that is discussed in detail in Chapter 9.

5.


Second Draw
First Draw
$\frac{6}{10}$
R
$\frac{5}{9}$
R
$\frac{4}{9}$
B
$\frac{4}{10}$
B
$\frac{6}{9}$
R
$\frac{3}{9}$
B

$P(0) = \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15}$

$P(1) = \frac{6}{10} \cdot \frac{4}{9} + \frac{4}{10} \cdot \frac{6}{9} = \frac{8}{15}$

$P(2) = \frac{6}{10} \cdot \frac{5}{9} = \frac{5}{15}$

Check: $P(0) + P(1) + P(2) = \frac{2}{15} + \frac{8}{15} + \frac{5}{15} = 1$

6. P(red on second) may be found by adding the probabilities of the branches R-R and B-R.

   $P(\text{red on second}) = \frac{6}{10} \cdot \frac{6}{10} + \frac{4}{10} \cdot \frac{6}{10} = \frac{36}{100} + \frac{24}{100} = \frac{60}{100} = \frac{3}{5}$

   Notice that $P(\text{red on second} \mid \text{red on first}) = \frac{3}{5}$. The events "red on second" and "red on first" are independent.

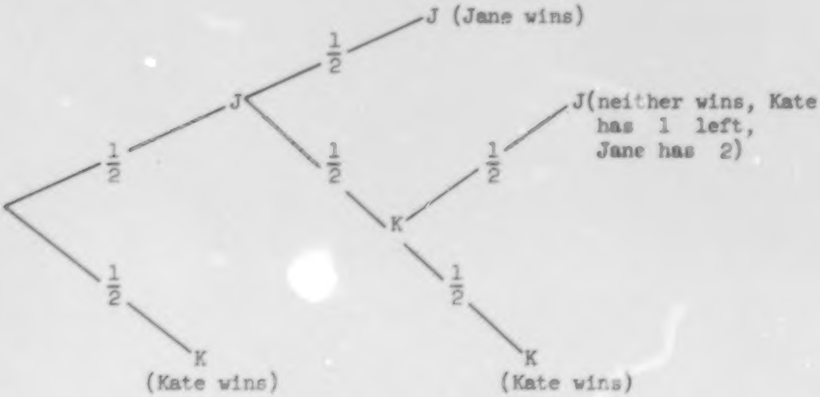
7. As in the answer to Exercise 6,

   $P(\text{red on second}) = \frac{6}{10} \cdot \frac{5}{9} + \frac{4}{10} \cdot \frac{6}{9} = \frac{30}{90} + \frac{24}{90} = \frac{54}{90} = \frac{3}{5}$. (!)

   Is it not surprising that P(red on second) is the same whether we draw with replacement or without replacement? In this case, "red on second" and "red on first" are not independent.

   $P(\text{red on second red on first}) = \frac{5}{9}$.

8.



$P(\text{Jane wins}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ (from branch "JJ")

$P(\text{Kate wins}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ (from branches "K" and "JKK")

$P(\text{neither wins}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$ (from branch "JKJ")

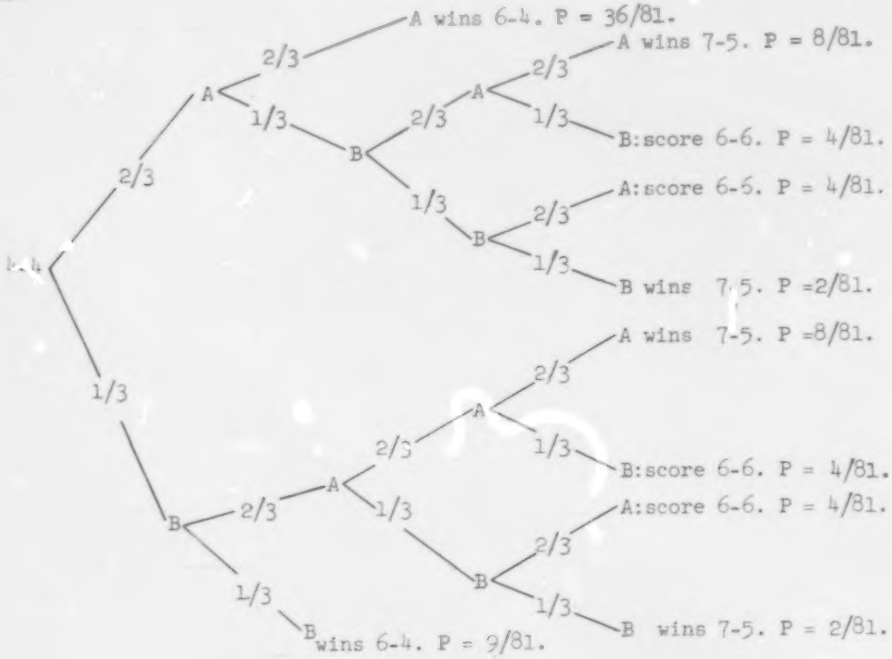
Check: $\frac{1}{4} + \frac{5}{8} + \frac{1}{8} = 1$

9. $P(0) = \frac{1}{4}$ (Jane wins)

$P(1) = \frac{1}{8}$ (branch "JKJ")

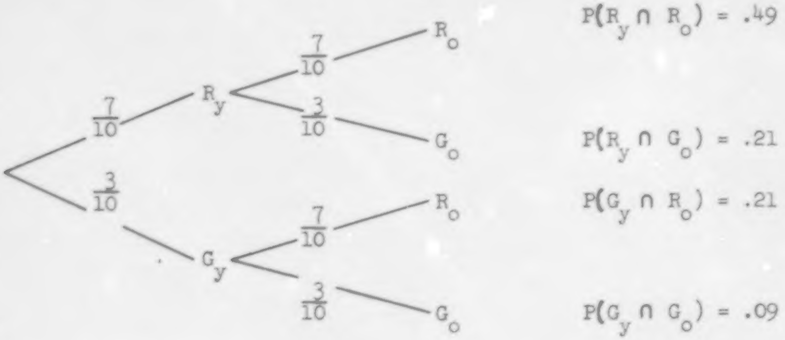
$P(2) = 0$
$P(3) = \frac{5}{8}$ (Kate wins)
The game can never end when Kate has exactly 2 chips. However, in say eight games, she would "expect" to have 0 chips twice, 1 chip once, and 3 chips 5 times. 0 + 1 + 15 = 16. Her "average" number of chips at the end of the game is 2.

10.


4-4
2/3
A
2/3
A wins 6-4. P = 36/81.
1/3
B
2/3
A
2/3
A wins 7-5. P = 8/81.
1/3
B:score 6-6. P = 4/81.
1/3
B
2/3
A:score 6-6. P = 4/81.
1/3
B wins 7-5. P =2/81.
1/3
B
2/3
A
2/3
A
2/3
A wins 7-5. P =8/81.
1/3
B:score 6-6. P = 4/81.
1/3
B
2/3
A:score 6-6. P = 4/81.
1/3
B wins 7-5. P = 2/81.
1/3
B wins 6-4. P = 9/81.

$P(\text{A wins}) = \frac{52}{81}$; $P(\text{B wins}) = \frac{13}{81}$: $P(\text{6-all}) = \frac{16}{81}$.

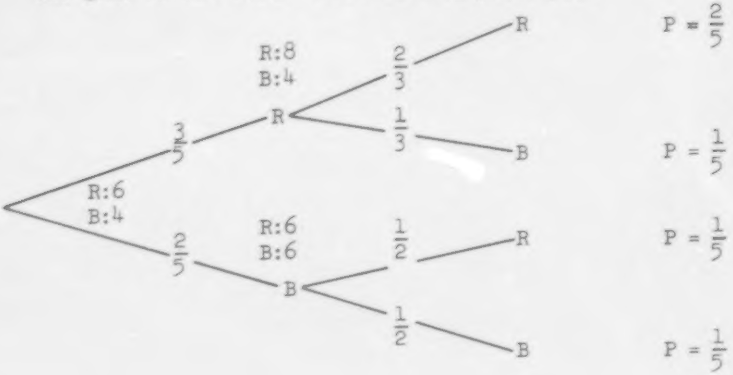
11.


$\frac{7}{10}$
R_y
$\frac{7}{10}$
R_o
$\frac{3}{10}$
G_o
$\frac{3}{10}$
G_y
$\frac{7}{10}$
R_o
$\frac{3}{10}$
G_o
$P(R_y \cap R_o) = .49$
$P(R_y \cap G_o) = .21$
$P(G_y \cap R_o) = .21$
$P(G_y \cap G_o) = .09$

$P(\text{win}) = P(R_y \cap R_o) + P(G_y \cap G_o) = .45 + .09 = .58$

The game is not fair. Does this surprise you?

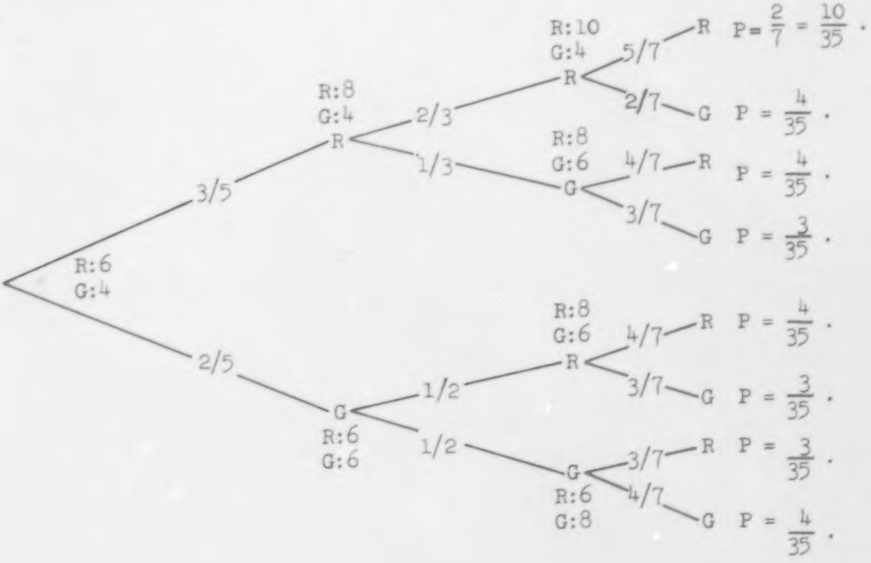
12.


R:6
B:4
$\frac{3}{5}$
R
R:8
B:4
$\frac{2}{3}$
R
$\frac{1}{3}$
B
$\frac{2}{5}$
B
R:6
B:6
$\frac{1}{2}$
R
$\frac{1}{2}$
B
$P = \frac{2}{5}$
$P = \frac{1}{5}$
$P = \frac{1}{5}$
$P = \frac{1}{5}$

$P(\text{red on second draw}) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$ (!)

Surprised? Refer back to Exercises 6 and 7.

13. (a)


R:10
G:4
5/7
R
$P = \frac{2}{7} = \frac{10}{35}$.
R:8
G:4
2/3
R
2/7
G
$P = \frac{4}{35}$.
R:8
G:6
1/3
4/7
R
$P = \frac{4}{35}$.
3/5
G
3/7
G
$P = \frac{3}{35}$.
R:6
G:4
R:8
G:6
4/7
R
$P = \frac{4}{35}$.
2/5
1/2
R
3/7
G
$P = \frac{3}{35}$.
G
R:6
G:6
1/2
3/7
R
$P = \frac{3}{35}$.
G
R:6
G:8
4/7
G
$P = \frac{4}{35}$.

$P(\text{red on third draw}) = \frac{10}{35} + \frac{4}{35} + \frac{4}{35} + \frac{3}{35} = \frac{21}{35} = \frac{3}{5}$.

(b) We hope you are led to guess that for this type of problem,

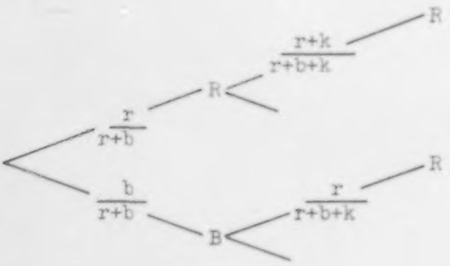
$P(\text{red on first draw}) = P(\text{red on second draw})$.

If we start with 5 red, 5 blue, then,

$P(\text{red on first draw}) = P(\text{red on second draw}) = \frac{1}{2}$.

If you are familiar with the manipulation of algebraic symbols you will understand the following argument.

Start with r red, b blue marble. Replace the marble drawn and add k of the same color. $P(\text{red on first draw}) = \frac{r}{r+b}$.



R
$\frac{r+k}{r+b+k}$
R
$\frac{r}{r+b}$
$\frac{b}{r+b}$
B
$\frac{r}{r+b+k}$
R

$$P = \frac{r}{r+b} \cdot \frac{r+k}{r+b+k}$$

$$P = \frac{b}{r+b} \cdot \frac{r}{r+b+k}$$

$$P(\text{red on second draw}) = \frac{r}{r+b} \cdot \frac{r+k}{r+b+k} + \frac{b}{r+b} \cdot \frac{r}{r+b+k}$$

$$= \frac{r^2+rk+rb}{(r+b)(r+b+k)}$$

$$= \frac{r(r+k+b)}{(r+b)(r+b+k)}$$

$$= \frac{r}{r+b} = P(\text{red on first draw})$$

Are you willing to guess further? How about the third drawing? The 100th drawing?

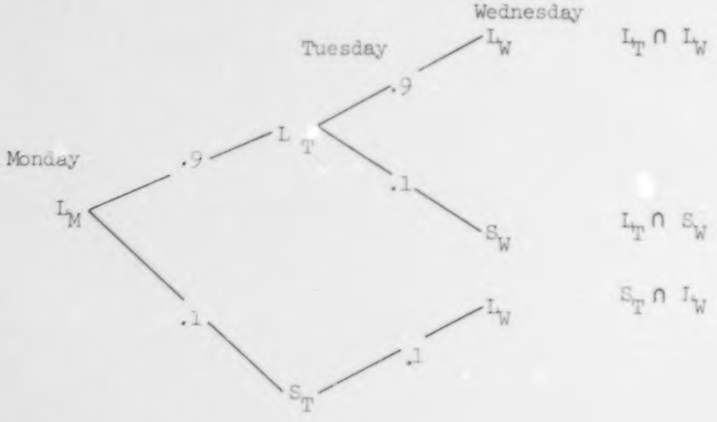
You might also refer back to Exercise 6. Here we replace but do not add marbles $(k=0)$.

In Exercise 7 we remove the marble drawn. $(k = -1)$

The result in all cases is

P(red on first draw) = P(red on second draw).

14.



Wednesday
Tuesday
L_W
$L_T \cap L_W$
.9
Monday
.9
L_T
.1
L_M
S_W
$L_T \cap S_W$
L_W
$S_T \cap L_W$
.1
.1
S_T

$P(S_W) = (.9)\,(.1) = .09$

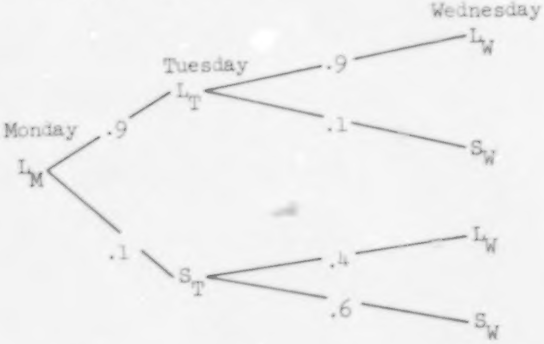
Note: $P(L_W) = (.9)(.9) + (.1)(.1) = .91$

†15. $P(\text{Burned}) = P(L_W \text{ and burned}) + P(S_W \text{ and burned})$

$$= P(L_W) \cdot P(\text{Burned} \mid L_W) + P(S_W) \cdot P(\text{Burned} \mid S_W)$$

$$= (.91)(.1) + (.09)(.4)$$
$$= .091 + .036 = .127$$

†16.


Monday
Tuesday
Wednesday
L_M
.9
.1
L_T
S_T
.9
.1
.4
.6
L_W
S_W
L_W
S_W

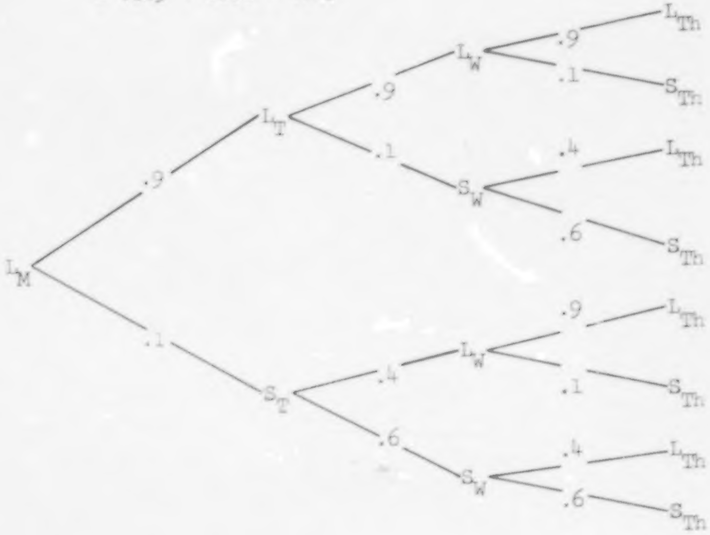
(a) $P(S_W) = (.9)(.1) + (.1)(.6) = .09 + .06 = .15$

Note: $P(L_W) = (.9)(.9) + (.1)(.4) = .81 + .04 = .85$

(b) $P(\text{Burned}) = P(L_W \text{ and burned}) + P(S_W \text{ and burned})$

$$= (.85)(.1) + (.15)(.4)$$
$$= .085 + .060 = .145$$

†17.


L_M
.9
.1
L_T
S_T
.9
.1
.4
.6
L_W
S_W
L_W
S_W
.9
.1
.4
.6
.9
.1
.4
.6
L_{Th}
S_{Th}
L_{Th}
S_{Th}
L_{Th}
S_{Th}
L_{Th}
S_{Th}

$$P(L_{Th}) = (.9)(.9)(.9) + (.9)(.1)(.4) + (.1)(.4)(.9) + (.1)(.6)(.4)$$
$$= .729 + .036 + .036 + .024 = .825$$
$$P(S_{Th}) = 1 - .825 = .175$$
So: $P(\text{Burned}) = (.825)(.1) + (.1) + (.175)(.4)$
$$= .0825 + .0700 = .1525$$

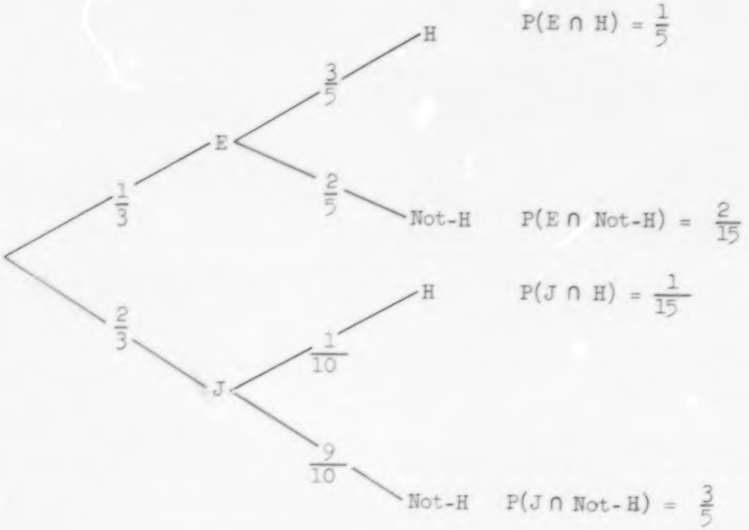
The probability that the cereal is burned has increased from .145 to .1525. This increase,however, is small (.0075). You might guess that future days the probability would again increase, but by smaller and smaller amounts. See Chapter 12.

Answers to Exercises 8-3

1. $$P(J \mid H) = \frac{P(J) \cdot P(H \mid J)}{P(E) \cdot P(H \mid E) + P(J) \cdot P(H \mid J)}$$
$$= \frac{\frac{2}{3} \cdot \frac{1}{10}}{\frac{1}{3} \cdot \frac{3}{5} + \frac{2}{3} \cdot \frac{1}{10}}$$
$$= \frac{\frac{1}{15}}{\frac{3}{15} + \frac{1}{15}} = \frac{\frac{1}{15}}{\frac{4}{15}} = \frac{1}{4}.$$

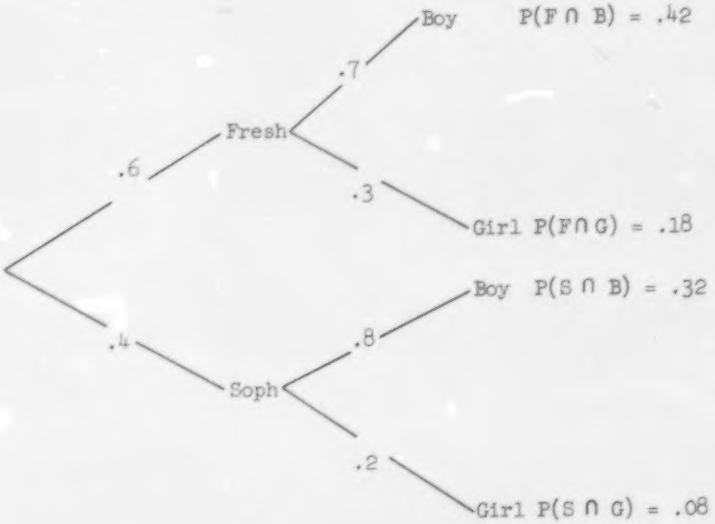
Of course, we knew that
$$P(J \mid H) = 1 - P(E \mid H)$$
$$\frac{1}{4} = 1 - \frac{3}{4}$$

2.



Answers to Exercises 8-4

1.


Boy P(F ∩ B) = .42
.7
Fresh
.6
.3
Girl P(F∩G) = .18
Boy P(S ∩ B) = .32
.4
.8
Soph
.2
Girl P(S ∩ G) = .08

(a) $P(G) = P(F \cap G) + P(S \cap G) = .18 + .08 = .26$

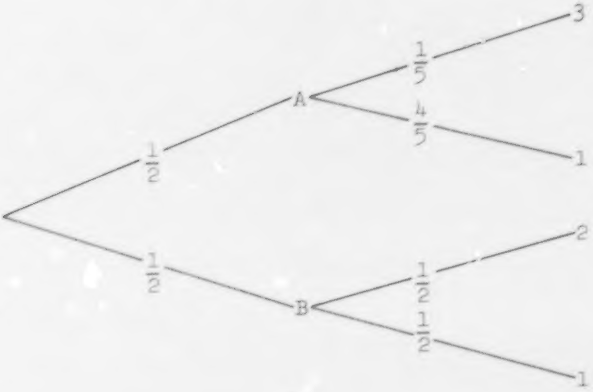
(b) $P(F|G) = \frac{P(F \cap G)}{P(G)}$

$$= \frac{.18}{.26} \sim .69$$

Notice that if we substitute directly into Bayes' formula we have:

$$P(F|G) = \frac{(.6)(.3)}{(.6)(.3) + (.4)(.2)} = \frac{.18}{.26} \sim .69$$

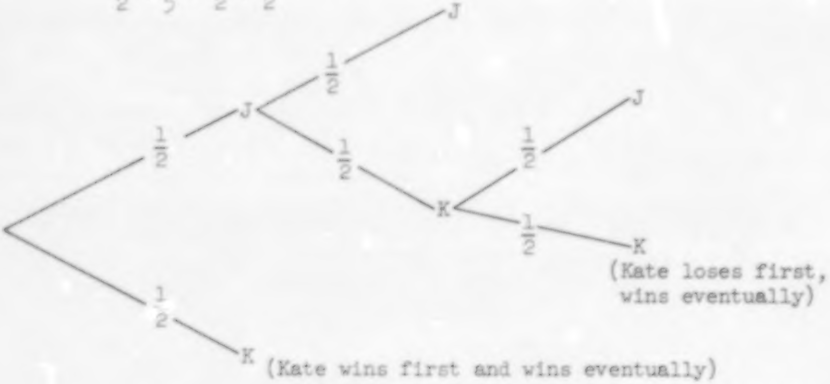
2.


3
1/5
A
4/5
1
1/2
1/2
2
1/2
B
1/2
1

$$P(A|1\text{ piece}) = \frac{P(A \cap 1)}{P(1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{4}{5}}{\frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{8}{13}$$

3.


J
1/2
1/2
J
J
1/2
1/2
K
1/2
1/2
K
(Kate loses first, wins eventually)
1/2
K
(Kate wins first and wins eventually)

$$P(K\text{ wins first } | \text{ K wins eventually}) = \frac{P(K\text{ wins first and wins eventually})}{P(K\text{ wins})}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{8}} = \frac{4}{5}$$

4. Refer to the answers to Exercise 1, Section 8-2 for the tree diagram.

$$P(II|2) = \frac{P(II \cap 2)}{P(2)}$$

$$= \frac{\frac{1}{3} \cdot \frac{1}{2}}{(\frac{1}{3} \cdot \frac{1}{3}) + (\frac{1}{3} \cdot \frac{1}{2}) + (\frac{1}{3} \cdot \frac{1}{2})} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8}$$

5. (a) With replacement. (Exercise 4, Section 8-2)

$$P(\text{ 1st red } | \text{ 2nd red}) = \frac{P(\text{two reds})}{P(\text{2nd red})}$$

$$= \frac{\frac{6}{10} \cdot \frac{6}{10}}{(\frac{6}{10} \cdot \frac{6}{10}) + (\frac{4}{10} \cdot \frac{6}{10})} = \frac{3}{5}$$

(b) Without replacement. (Exercise 5, Section 8-2)

$$P(\text{ 1st red } | \text{ 2nd red }) = \frac{P(\text{two reds})}{P(\text{2nd red})}$$

$$= \frac{\frac{6}{10} \cdot \frac{5}{9}}{(\frac{6}{10} \cdot \frac{5}{9}) + (\frac{4}{10} \cdot \frac{6}{9})} = \frac{5}{9}$$

(c) With replacement and add two of same color. (Exercise 12, Section 8-2)

$$P(\text{1st red} \mid \text{2nd red}) = \frac{P(\text{two reds})}{P(\text{2nd red})}$$

$$= \frac{\frac{3}{5} \cdot \frac{2}{3}}{(\frac{3}{5} \cdot \frac{2}{3}) + (\frac{2}{5} \cdot \frac{1}{2})} = \frac{2}{3}$$

Actually, you did not need to perform these calculations if you had remembered the answer to the earlier exercises. Recall that:

$$P(\text{1st red} \mid \text{2nd red}) = \frac{P(\text{two reds})}{P(\text{2nd red})}$$

and $$P(\text{2nd red} \mid \text{1st red}) = \frac{P(\text{two reds})}{P(\text{1st red})}$$

The numerators of the two fractions are the same. Also, we learned in the exercises of Section 8-2 that $P(\text{2nd red}) = P(\text{1st red}) = \frac{3}{5}$ for all three urn problems. So

$$P(\text{1st red} \mid \text{2nd red}) = P(\text{2nd red} \mid \text{1st red})$$

But $P(\text{2nd red} \mid \text{1st red})$ is easy to find;

(a) with replacement: after drawing red, we have left 6 red marbles out of 10, so

$$P(\text{2nd red} \mid \text{1st red}) = \frac{3}{5},$$

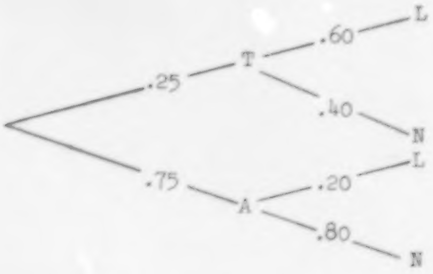
(b) without replacement: after drawing red, we have left 5 red marbles out of 9, so

$$P(\text{2nd red} \mid \text{1st red}) = \frac{5}{9},$$

(c) with replacement and adding two: after drawing red we are left with 8 red marbles out of 12, so

$$P(\text{2nd red} \mid \text{1st red}) = \frac{2}{3}.$$

6. Using the notation of the answer to Exercise 8, Section 7-4, we construct a tree and substitute in Bayes' formula

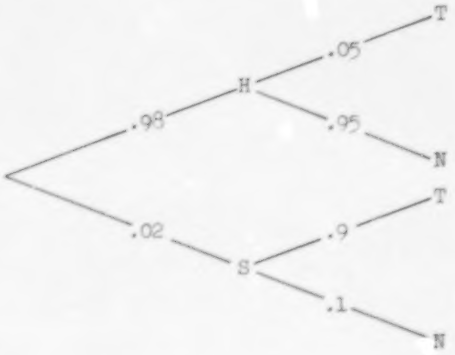

.25
T
.60
L
.40
N
.75
A
.20
L
.80
N

$$P(A \mid L) = \frac{P(A)\cdot P(L|A)}{P(A)\cdot P(L|A)+P(T)\cdot P(L|T)}$$

$$= \frac{(.75)(.20)}{(.75)(.20)+(.25)(.60)}$$

$$= \frac{.15}{.15+.15} = .50$$

7. Using the notation of the answer Exercise 9, Section 7-4, we construct a tree diagram and substitute in Bayes' formula.


.98
H
.05
T
.95
N
.02
S
.9
T
.1
N

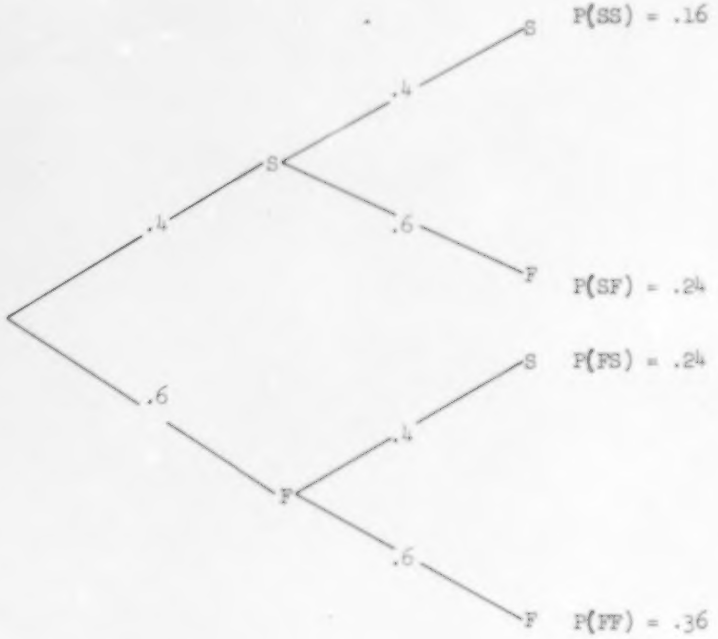
$$P(H \mid T) = \frac{P(H)\cdot P(T \mid H)}{P(H)\cdot P(T \mid H) + P(S)\cdot P(T \mid S)}$$

$$= \frac{(.98)(.05)}{(.98)(.05) + (.02)(.90)}$$

$$= \frac{.049}{.049 + .018} = \frac{.049}{.067} \sim .73$$

Answers to Exercises 9-2

1.

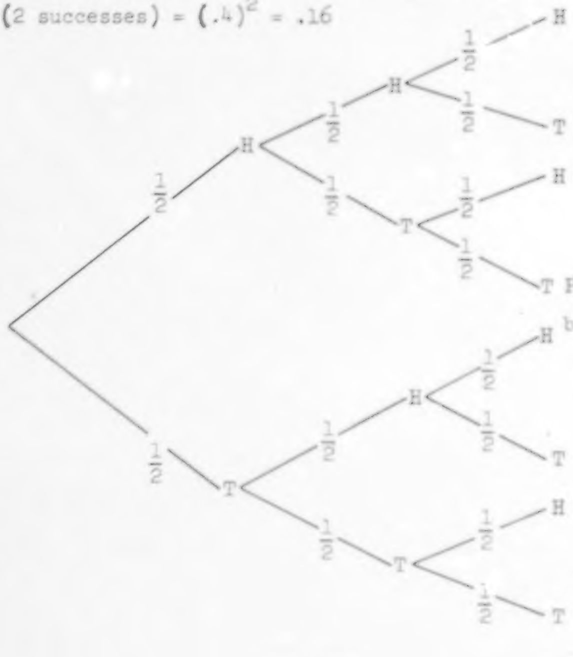

.4
.6
S
F
.4
.6
.4
.6
S
F
S
F
P(SS) = .16
P(SF) = .24
P(FS) = .24
P(FF) = .36

P (0 successes) = $(.6)^2$ = .36

P (1 success) =2(.4)(.6) = .48 (There are two branches involved)

P (2 successes) = $(.4)^2$ = .16

2.


$\frac{1}{2}$
H
T
H
T
H
T
H
T
H
T
H
T
Probability of each branch is:
$(\frac{1}{2})^3 = \frac{1}{8}$

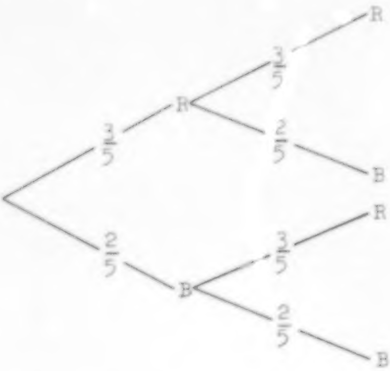
P (0 heads) = $\frac{1}{8}$

P (1 head) = $\frac{3}{8}$ (Three branches!)

P (2 heads) = $\frac{3}{8}$

P (3 heads) = $\frac{1}{8}$

3.


R
3/5
R
2/5
B
3/5
R
2/5
B
3/5
R
2/5
B

P(RR) = $\frac{9}{25}$

P(RB) = $\frac{6}{25}$

P(BR) = $\frac{6}{25}$

P(BB) = $\frac{4}{25}$

P (0 reds) = $\frac{4}{25}$

P (1 red) = $2 \cdot \frac{6}{25} = \frac{12}{25}$

P (2 reds) = $\frac{9}{25}$

Answers to Exercises 9-3

1. $P(\text{no hits}) = (.7)^4$, or approximately .24 .

   $P(1\ \text{hit}) = 4(.3)(.7)^3$, or approximately .41 .

   $P(2\ \text{hits}) = 6(.3)^2(.7)^2$, or approximately .26 .

   $P(3\ \text{hits}) = 4(.3)^3(.7)$, or approximately .08 .

   $P(4\ \text{hits}) = (.3)^4$, or approximately .01 .

   He makes at least two hits if his number of hits is 2, 3, or 4. Since exactly 2 hits, exactly 3 hits, and exactly 4 hits are mutually exclusive events, we have: $P(\text{at least } 2 \text{ hits}) = .26 + .08 + .01$, or .35 .

2. $5(\frac{1}{6})(\frac{5}{6})^5 \approx .34$.

3. (a)

Number of heads	0	1	2	3	4
Probability	.06	.25	.38	.25	.06

   (c) The most likely number of heads is 2. The probability of getting exactly 2 heads is .38. (We have rounded to 2 decimals.)

4. $P(\text{no } 6\text{'s}) = (\frac{5}{6})^{10} \approx .162$

   $P(\text{at least one } 6) = 1 - (\frac{5}{6})^{10} \approx .838$

Answer to Exercise 9-5

Our record of 100 throws gave these results:

Number of Evens	0	1	2	3	4	5
Number of times occurring	1	5	7	5	2	0
Fraction of times occurring	.05	.25	.35	.25	.10	.00

Our values, .05, .25, .35, .25, .10, .00 are reasonably close to the tabulated values: .03, .16, .31, .31, .16, .03.

Answers to Exercises 9-6

1. $x^7 + 7x^6y + 21x^5y^2$.

2. $1 + 5(.01) + 10(.01)^2 + 10(.01)^3 + 5(.01)^4 + (.01)^5 = 1.0510100501$.

3. $(1.02)^6 = 1 + 6(.02) + 15(.02)^2 + 20(.02)^3 + 15(.02)^4 + 6(.02)^5 + (.02)^6$.

   Since $(.02)^3 = .000008$, each of the 4 last terms is less than $20(.000008)$, or $.00016$. Hence their sum is less than $.00064$. To the nearest hundredth,

$$(1.02)^6 \approx 1 + .12 + .006,$$

$$(1.02)^6 \approx 1.13$$

4. (a) $x^3 + 6x^2y + 12xy^2 + 8y^3$; $[x^3 + 3x^2(2y) + 3x(2y)^2 + (2y)^3]$

   (b) $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

   (c) $x^5 + 5$ [illegible] $\frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$

   (d) $x^3 + \frac{3}{2}x^2 + \frac{3}{4}x + \frac{1}{8}$

5. (a) $(x + y)^5 = \binom{5}{5}x^5 + \binom{5}{4}x^4y + \binom{5}{3}x^3y^2 + \binom{5}{2}x^2y^3 + \binom{5}{1}xy^4 + \binom{5}{0}y^5$.

   (b) $32 = 2^5 = (1 + 1)^5 = \binom{5}{5} \cdot 1^5 + \binom{5}{4} 1^4 \cdot 1 + \binom{5}{3}1^3 \cdot 1^2 + \binom{5}{2}1^2 \cdot 5^3 +$

$$\binom{5}{1}1^1 \cdot 1^4 + \binom{5}{0} \cdot 1^5 = \binom{5}{5} + \binom{5}{4} + \binom{5}{3} + \binom{5}{2} + \binom{5}{1} + \binom{5}{0}$$

   (c) $2^n = (1 + 1)^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1} + \binom{n}{0}$

   Of course, we already knew this result, from our consideration of counting outcomes -- from tree diagrams, for example.

Answers to Exercises 9-8

1. 2 R and no G : $\frac{10}{243}$

   2 R and one G : $\frac{30}{243}$

   2 R and 3 G : $\frac{10}{243}$

2. P(exactly 2 R) is the sum of the three probabilities found in Ex. 1 and the probability of 2 R and 2 G found in Item 9.

$$\frac{10}{243} + \frac{30}{243} + \frac{10}{243} + \frac{30}{243} = \frac{80}{243}$$

3. P(exactly 2 R) can also be found by the method of Sec. 9-2. We have 5 independent trials, and for each the probability of spinning red is $\frac{1}{3}$ while the probability of not-red is $\frac{2}{3}$.

There are 10 words in R,N with exactly 2 R's.

Hence

$$\begin{aligned} P(\text{exactly } 2 \text{ R}) &= 10\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 \\ &= \frac{80}{243} \ . \end{aligned}$$

5. (a) $x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$;

(b) $x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3xz^2 + 3yz^2 + 3x^2z + 3y^2z + 6xyz$.

6. $4x^3y$, $6x^2y^2$, $12xyz^2$.

7. $\frac{4}{81}$, $\frac{6}{81}$, $\frac{12}{81}$.

Answers to Exercises 10-2

1. Our average was .9.

2. (a) For 100 throws of a die, our average was:

$[12(1) + 15(2) + 25(3) + 15(4) + 21(5) + 12(6)]\left(\frac{1}{100}\right)$, or 3.54 .

(b) For the first 25 throws, our average was:

$[4(1) + 1(2) + 6(3) + 6(4) + 5(5) + 3(6)]\left(\frac{1}{100}\right)$,

or 3.64.

(c) For the last 25 throws, our average was:

$[3(1) + 8(2) + 4(3) + 2(4) + 5(5) + 3(6)]\left(\frac{1}{100}\right)$, or 3.28 .

Your averages, of course, are not likely to be exactly the same.

3. (a) $\frac{1}{10}(2.8) + \frac{3}{10}(2.9) + \frac{2}{5}(3.0) + \frac{1}{5}(3.1) = 2.97$

The average is 2.97 inches.

(b) 100 strips

(c) 1 (This is what you would expect, if the data in the problem is complete.)

4. He has lost 39 balls. (2.6 × 15)

Answers to Exercises 10-3

1. (a)

Value of prize in dollars	1	.6	.2
Probability	.3	.45	.25

(b) $M = 1(.3) + .6(.45) + .2(.25) = .62$

2. (a)

Number spaces	1	2	3
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(b) $M = \frac{9}{4} = 2\frac{1}{4}$

3. (a) $\frac{1}{2}$

(b) 1

(c) 2. **Note, we had** $M = \frac{1}{2}$ for 1 coin

$M = 1$ for 2 coins

$M = \frac{3}{2}$ for 3 coins.

(You might wish to verify your guess. You will find that 2 is the correct value.)

4. $M = 0(.94) + 1(.03) + 2(.02) + 3(.01)$,

$M = 0.03 + 0.04 + 0.03 = 0.10$,

$M = 0.10$.

It can be expected that in 100 such periods, 10 accidents might occur.

Answers to Exercises 10-4

1. In each part of the problem, the king should compute the mathematical expectation of his loss if the army is at A and if it is at B. He should then choose the alternative for which the loss is smallest.

(a) The losses, with probabilities, and the expected value:

Army at A :

	enemy attacks A	enemy attacks B
loss	1	2
probability	$\frac{1}{2}$	$\frac{1}{2}$

$M = 1(\frac{1}{2}) + 2(\frac{1}{2}) = \frac{3}{2}$

Army at B :

	enemy attacks A	enemy attacks B
loss	4	0
probability	$\frac{1}{2}$	$\frac{1}{2}$

$M = 4(\frac{1}{2}) + 0(\frac{1}{2}) = 2$

The expected value of the loss is smaller when the army is at A. The king should send the army to A , since he wants the loss to be as small as possible.

(b) In this case, P(enemy attacks A) = $\frac{1}{5}$, P(enemy attacks B) = $\frac{4}{5}$.

Army at A :

	enemy attacks A	enemy attacks B
loss	1	2
probability	$\frac{1}{5}$	$\frac{4}{5}$

$M = 1(\frac{1}{5}) + 2(\frac{4}{5}) = \frac{9}{5}$

Army at B :

	enemy attacks A	enemy attacks B
loss	4	0
probability	$\frac{1}{5}$	$\frac{4}{5}$

$M = 4(\frac{1}{5}) + 0(\frac{4}{5}) = \frac{4}{5}$

The expected loss is now smaller when the army is at B. The king should send his army to B , since he wants his loss to be as small as possible.

(c) P(enemy attacks A) = $\frac{2}{5}$, P(enemy attacks B) = $\frac{3}{5}$.

Army at A :

	enemy attacks A	enemy attacks B
loss	1	2
probability	$\frac{2}{5}$	$\frac{3}{5}$

$$M = 1(\tfrac{2}{5}) + 2(\tfrac{3}{5}) = \frac{8}{5}$$

Army at B :

	enemy attacks A	enemy attacks B
loss	4	0
probability	$\frac{2}{5}$	$\frac{3}{5}$

$$M = 4(\tfrac{2}{5}) + 0(\tfrac{1}{5}) = \frac{8}{5}$$

The expected values are equal: It does not matter whether the army is at A or at B . But, of course, he will still send his army to one place or the other. He might, perhaps, toss a coin to decide where to send it.

2. (a)

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(If you had difficulty, refer to Chapter 3.)

(b) $M = 7$.

3. (a) Since there is no reason to feel that one die has a special "advantage", it is reasonable to guess that the difference is 0 .

(b) Probability table for difference:

Difference	-5	-4	-3	-2	-1	0	1	2	3	4	5
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$M = 0$.

4. (a) In both games you are multiplying two numbers. Hence you might feel that the two players can expect to win the same amounts. But this is not true, as your results in (b) show you.

(b) For the product of the two numbers thrown, the table of probabilities is:

Product	1	2	3	4	5	6	8	9	10	12	15	16	18	20	24	25	30	36
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$M = 12\frac{1}{4}$

For the square of the numbers on red:

Square	1	4	9	16	25	36
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$M = 15\frac{1}{6}$

Answers to Exercises 10-5

1. (a) P (no match) = 0

   P (one match) = 1

   (b) Expected value of number of matches = 1.

2. (a) P (no match) = $\frac{1}{2}$

   P (one match) = 0

   P (2 matches) = $\frac{1}{2}$

   (b) Expected value = 1 . $[0(\frac{1}{2}) + 1(0) + 2(\frac{1}{2})]$

3. (a) P (no match) = $\frac{2}{6} = \frac{1}{3}$

   P (one match) = $\frac{3}{6} = \frac{1}{2}$

   P (2 matches) = 0

   P (3 matches) = $\frac{1}{6}$

   If you had trouble, simply write the 6 possible ways of arranging 3 cards, and count the number of matches for each arrangement.

   (b) Expected value = 1

4. If you couldn't guess a generalization, look again at 1 (a), 2 (a), 3 (a). Where do the 0's occur? Then look at 1 (b), 2 (b), 3(b). You could guess:

   (a) It can never happen that all except 1 card matches. With 6 cards, for example, you can't have exactly 5 cards in proper position. A little thought will convince you that this generalization is correct.

   (b) It appears that the expected value is 1 in every case.

Answers to Exercises 10-6

1. For a single trial, we have

Number of successes	0	1
Probability	q	p

Expected value = $0(q) + 1(p) = p$.

For n trials, we add the expected value for each trial, attaining np .

2. (a) Expected value of number of success on each trial = $\frac{1}{2}$.

Number of successes on 4 trials = sum of number of successes on each trial.

Expected value of number of successes = $4(\frac{1}{2}) = 2$.

(b) Most probable number of successes = 2. This can be found directly from the graph. The greatest probability (.38) is associated with 2.

(c) 1 , 2 , 3 differ from 2 by at most 1 . The probability that the number of successes is 1 , 2 or 3 is found by adding the separate probabilities. .25 + .38 + .25 = .88.

			(a)	(b)	(c)
Ex.	n	p	np	Most probable number of successes	Probability that number of successes differs from np by at most 1.
3.	4	$\frac{1}{3}$	$\frac{4}{3}$	1	.70
4.	4	.8	3.2	3 , 4 equally likely	.82
5.	4	.2	.8	0 , 1 equally likely	.82
6.	4	.6	2.4	2 , 3 equally likely	.70
7.	6	$\frac{1}{2}$	3	3	.77
8.	8	$\frac{1}{2}$	4	4	.71
9.	10	$\frac{1}{2}$	5	5	.67

10. (a) $P(\text{red} \mid \text{I}) = \frac{1}{2}$, since the red I area is half the I area.

$P(\text{I} \mid \text{red}) = \frac{2}{5}$.

The red region is $\frac{5}{8}$ of the entire circular region. The red I region is thus $\frac{2}{5}$ of the red region.

$P(\text{II} \mid \text{red}) = \frac{3}{5}$.

(b) For this spinner, the probabilities are exactly the same as $P(\text{red} \mid \text{I})$, $P(\text{I} \mid \text{red})$, etc., for the two spinners at the beginning of the section. Use this spinner if you had trouble with the probabilities for the example.

Answers to Exercises 10-7

1. Average height = 65 inches

   Average weight $\approx$ 120.3 lb.

2. Standard deviation of heights $\approx$ 4.3 inches

Answers to Exercises 10-8

1. $M = 1.48$

   Standard deviation is approximately .77 .

2.

Number of points x	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$(x - M)^2$	$(1-3.5)^2$ $=(-2.5)^2$	$(2-3.5)^2$ $=(-1.5)^2$	$(3-3.5)^2$ $= (-0.5)^2$	$(4-3.5)^2$ $=(0.5)^2$	$(5-3.5)^2$ $=(1.5)^2$	$(6-3.5)^2$ $=(2.5)^2$

$$6.25(\tfrac{1}{6}) + 2.25(\tfrac{1}{6}) + .25(\tfrac{1}{6}) + .25(\tfrac{1}{6}) + 2.25(\tfrac{1}{6}) + 6.25(\tfrac{1}{6})$$

$$= 17.50(\tfrac{1}{6}) = \frac{35}{12}, \text{ or approximately } 2.92 .$$

Standard deviation = $\sqrt{2.92}$, or approximately 1.71 .

3. Our results for our first 25 throws of a die:

   $M = 3.64$ (Note that this is a good estimate of the expected value, 3.5).

   Standard deviation is approximately 1.55 .

Answers to Exercises 10-9

1. We have:

Number heads	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$E(N) = \frac{3}{2}$

$E(N^2) \quad 0^2(\frac{1}{8}) + (1^2)(\frac{3}{8}) + (2^2)(\frac{3}{8}) + (3^2)(\frac{1}{8}) = \frac{24}{8} = 3$

$E(N^2) - (E(N))^2 = 3 - (\frac{3}{2})^2 = \frac{3}{4}$

Standard deviation = $\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$.

2.

Number heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$E(N) = 2$

$E(N^2) = 0(\frac{1}{16}) + 1(\frac{4}{16}) + (2^2)(\frac{6}{16}) + (3^2)(\frac{4}{16}) + (4^2)(\frac{1}{16})$

$= \frac{4 + 24 + 36 + 16}{16} = \frac{80}{16} = 5$

$E(N^2) - (E(N))^2 = 5 - 4 = 1$

Standard deviation = $\sqrt{1} = 1$

3.

Numbers 1's	0	1	2	3
Probability	$(\frac{5}{6})^3$	$3(\frac{1}{6})(\frac{5}{6})^2$	$3(\frac{1}{6})^2(\frac{5}{6})$	$(\frac{1}{6})^3$

$E(N^2) = 0^2 \ (\frac{5}{6})^3 + 1^2 \cdot 3(\frac{1}{6})(\frac{5}{6})^2 + 2^2 \cdot 3(\frac{1}{6})^2(\frac{5}{6}) + 3^2 \ (\frac{1}{6})^3$

$= \frac{75 + 60 + 9}{6^3} = \frac{144}{6^3} = \frac{2}{3}$

$E(N^2) - (E(N))^2 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ (Recall $E(N) = np = 3\ (\frac{1}{6}) = \frac{1}{2}$)

Standard deviation = $\sqrt{\frac{5}{12}} = \frac{1}{6}\sqrt{15}$

14.

Number of successes	0	1	2	3
Probability	q^3	$3pq^2$	$3p^2q$	p^3

$E(N) = 3p$

$$E(N^2) = 0^2 \cdot q^3 + 1^2 \cdot 3pq^2 + 2^2 \cdot 3p^2q + 3^2\, p^3$$

$$= 3pq^2 + 12p^2q + 9p^3$$

$$E(N^2) - (E(N))^2 = 3pq^2 + 12p^2q + 9p^3 - 9p^2$$

$$= 3pq^2 + 12p^2q + 9p^2\,(p - 1)\ .$$

In the last term, we can replace $p - 1$ by $-q$, since $p + q = 1$.

We have:

$$E(N^2) - (E(N))^2 = 3pq^2 + 12p^2q - 9p^2q$$

$$= 3pq^2 + 3p^2q$$

$$= 3pq\,(q + p)$$

$$= 3pq \quad (\text{since } p + q = 1)$$

Standard deviation = $\sqrt{3pq}$

Answers to Exercises 10-10

1. From the data, we can get the distribution for the number of games in a series.

Number of games	4	5	6	7
Frequency	7	7	8	14

$m = (4 \cdot 7 + 5 \cdot 7 + 6 \cdot 8 + 7 \cdot 14)\,\frac{1}{36}\ ,$

$m \sim 5.81\ .$

2. A team must win 4 games and, if more than 4 games are played, the winning team must, of course, win the last one, the decisive one. So, we have to consider four cases:

Case	No. of games played	The winning team wins	The losing team wins
(1)	4	all four	none
(2)	5	3 out of 4 and the 5th	1 of first 4
(3)	6	3 out of 5 and the 6th	2 of first 5
(4)	7	3 out of 6 and the 7th	3 out of first 6

For each of these cases, we must take into account the possibility that either team might be the series winner.

Let us write $P(B4)$, $P(B5)$, $P(B7)$, etc., to indicate the probability that the Blue Sox win in four games, five games, seven games, etc.

(1) What is the probability that the Blue Sox win in 4 games?

$$P(B4) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} .$$

$$P(B4) = \left(\frac{2}{3}\right)^4 \sim \underline{\underline{0.20}} .$$ Check the arithmetic!

The probability that the Blue Sox win in 4 games is 0.20 .

(2) What is the probability that the Blue Sox win in 5 games?

Remember: In this case, the Blue Sox must win the fifth game!
The Green Sox could win the 1st, 2nd, 3rd, or 4th game. So we have the 4 possibilities:

$\text{GBBB}\underline{\text{B}}$, $\text{BGBB}\underline{\text{B}}$, $\text{BBGB}\underline{\text{B}}$, $\text{BBBG}\underline{\text{B}}$.

(Note that the number of possibilities is $\binom{4}{1} = 4$. The probability of any one of these possibilities is $\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)$.)

So we can write:

$$P(B5) = \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) \cdot 4$$

$$= \left(\frac{2}{3}\right)^4\left(\frac{4}{3}\right)$$

$$\approx (0.20)(1.33) ,$$

$$P(B5) \approx \underline{\underline{0.26}} .$$

(Note: We have already found $\left(\frac{2}{3}\right)^4$ in (1) above .)

The probability that the Blue Sox win in 5 games is 0.26 .

(3) What is the probability that the Blue Sox win in 6 games?

This time the Green Sox win any two of the five games. The series may be played in 10 different ways:

$\binom{5}{2} = \frac{5 \cdot 4}{1 \cdot 2} = 10$. (List the possibilities if you weren't sure.)

So we have:

$$P(B6) = \left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^2(10)$$

$$P(B6) = \left(\frac{2}{3}\right)^4\left(\frac{10}{9}\right) \sim (0.20)(1.11)$$

$$P(B6) \sim \underline{\underline{0.22}} .$$

The probability that the Blue Sox win in 6 games is 0.22 .

(4) What is the probability that the Blue Sox win in 7 games?

Now the Green Sox win any three of the first six games, so there are 20 ways $\left(\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20\right)$ in which the Blue Sox can win the series. As a result we can write:

$$P(B7) = (\tfrac{2}{3})^4(\tfrac{1}{3})^3(20)$$

$$P(B7) = \left((\tfrac{2}{3})^4(\tfrac{10}{9})\right)(\tfrac{2}{3}) \sim (0.22)(0.67) ,$$

$$P(B7) \sim \underline{\underline{0.15}} .$$

Note again how we were able to simplify the computation by using the result of (3) .

Next we must compute the probabilities for the Green Sox:

What are the chances that the Green Sox win in four, five, six or seven games?

You should be able to follow the four cases quite easily without further explanation:

(a) What is the probability that the Green Sox win **in 4 games?**

$$P(G4) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \sim \underline{\underline{0.01}} . \qquad \text{(Compare with } P(B4) \text{ .)}$$

(b) What is the probability that the Green Sox win in 5 games?

$$P(G5) = (\tfrac{1}{3})^4(\tfrac{2}{3}) \cdot 4 . \qquad \text{(Compare with } P(B5) \text{ .)}$$

$$P(G5) = (\tfrac{1}{3})^4(\tfrac{8}{3}) \sim \underline{\underline{0.03}} .$$

(c) What is the probability that the Green Sox win in 6 games?

$$P(G6) = (\tfrac{1}{3})^4(\tfrac{2}{3})^2\binom{5}{2} . \qquad \text{(Compare with } P(B6) \text{ .)}$$

$$P(G6) = (\tfrac{1}{3})^4(\tfrac{8}{3})(\tfrac{5}{3}) \sim (0.03)(1.67) \sim \underline{0.05} .$$

(d) What is the probability that the Green Sox win in 7 games?

$$P(G7) = (\tfrac{1}{3})^4(\tfrac{2}{3})^3\binom{6}{3} . \qquad \text{(Compare with } P(B7) \text{ .)}$$

$$P(G7) = (\tfrac{1}{3})^4(\tfrac{8}{3})(\tfrac{5}{3})(\tfrac{4}{3}) \sim (0.05)(1.33) \sim \underline{\underline{0.07}} .$$

We can state the result: The probability that the Blue Sox win in

4 games is 0.20,
5 games is 0.26,
6 games is 0.22,
7 games is 0.15.

The probability that the Blue Sox win in 4, 5, 6, or 7 games is

$$0.20 + 0.26 + 0.22 + 0.15 = \underline{\underline{0.83}} .$$

In other words, the probability that the Blue Sox win the Series is 0.83.

The probability that the Green Sox win in

4 games is 0.01,
5 games is 0.03,
6 games is 0.05,
7 games is 0.07 .

The probability that the Green Sox win in 4, 5, 6, or 7 games is

$$0.01 + 0.03 + 0.05 + 0.07 = \underline{\underline{0.16}} .$$

The probability that the Green Sox win the Series is 0.16 .

The series will end in 4 games, if the Blue Sox or the Green Sox win in 4 games:

$$P(B \text{ or } G, 4) = 0.20 + 0.01 = \underline{0.21} .$$

Similarly,
$$P(B \text{ or } G, 5) = 0.26 + 0.03 = \underline{0.29} ,$$
$$P(B \text{ or } G, 6) = 0.22 + 0.05 = \underline{0.27} ,$$
$$P(B \text{ or } G, 7) = 0.15 + 0.07 = \underline{0.22} .$$

Note: Either the Blue Sox or the Green Sox win the series, so we have

$$P(B \text{ or } G) = 1 \;:\; 0.83 + 0.16 = 0.99 .$$

Can you account for the result of 0.99 instead of 1 ?

What is the expected number of games in the World Series?

No. of Games	4	5	6	7
P(B or G wins)	0.21	0.29	0.27	0.22

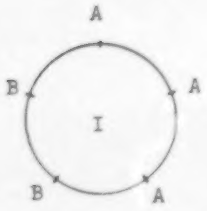
$$M = 4(0.21) + 5(0.29) + 6(0.27) + 7(0.22) ,$$
$$M = 0.84 + 1.45 + 1.62 + 1.54 ,$$
$$M = 5.45 .$$

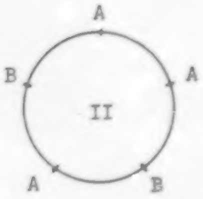
The expected number of games is 5.45 .

Answers to Exercises 11-3

1. $\binom{5}{2}$, or 10 .

2. We cannot be sure, of course, which ordering you chose. However, we can be sure that your circle matches one or the other (but not both) of the circles below.





Circle I	Circle II
A A A B B	A A B A B
A A B B A	A B A B A
A B B A A	B A B A A
B B A A A	A B A A B
B A A A B	B A A B A

Your list should match one of these two.

3. For "A always leads" we have:

Circle I	Circle II
A A A B B	A A B A B

4. and 5. Your first list of orderings (Exercise 2) did not contain all the possible orderings. If your circle was like I, for example, then you had only the orderings for it. Your circle for Exercise 4 again matches one of ours. In fact, it matches the one you did not use for Exercise 2.

6. P(A always leads) = $\frac{2}{10} = \frac{1}{5}$. (There are 2 "always leads" orderings out of 10 possible orderings. All the orderings are equally likely.)

7. 1.

8. Yes. $\frac{5 - 0}{5 + 0} = 1$.

9. Yes. P(A always leads) = $\frac{2 - 1}{2 + 1} = \frac{1}{3}$. Check that this is the correct result by listing the possible orderings for a 2 to 1 vote.

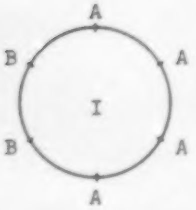
10. We do not know how the 14A's and 11B's are arranged on your circle. However, we are sure that in your experiment, as in ours, there were exactly 3 starting points which gave orderings where A always leads. Did you happen to notice that $\frac{14 - 11}{14 + 11} = \frac{3}{25}$?

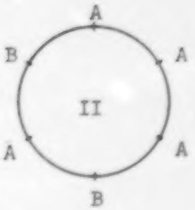
Answers to Exercises 11-5

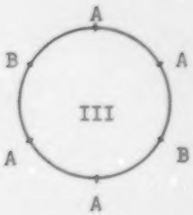
1. $\frac{7 - 3}{7 + 3} = \frac{4}{10} = \frac{2}{5}$.
2. At 1, 3, 6, or 7 .
3. 1, 4, 9.
4. Start at 4, 5, 6, or 1 .
5. $\frac{16 - 8}{16 + 8} = \frac{8}{24} = \frac{1}{3}$.
6. $\frac{4 - 3}{4 + 3} = \frac{1}{7}$.
7. $\frac{6 - 4}{6 + 4} = \frac{2}{10} = \frac{1}{5}$.
8. 0. A tennis set does not go to 42 games unless there have been several ties. At each tie, Tilden was not ahead.
9. $\frac{13 - 9}{13 + 9} = \frac{4}{22}$ or $\frac{2}{11}$. Poor king!
10. In this case we must count orderings in which A always leads or ties. You have already studied all the possible orderings for the 3 to 2 case. (Exercises 1 to 4 , Section 11-3.) You can find directly -- by counting: P(reaches goal) = $\frac{1}{2}$.
11. $\frac{2 - 1}{2 + 1} = \frac{1}{3}$.

Answers to Exercises 11-6

1. $\binom{6}{2}$, or 15 .
2. Your circles can be matched, by proper choice of starting position, with these:







A A A A B B
A A A B B A
A A B B A A
A B B A A A
B B A A A A
B A A A A B

A A A B A B
A A B A B A
A B A B A A
B A B A A A
A B A A A B
B A A A B A

A A B A A B
A B A A B A
B A A B A A

Did you notice? Circle III does not give 6 different orderings.

The 6 different starting positions yield only 3 different orderings.

Answers to Exercises 12-2

1. P(cooks L on Thursday) = .909 (See Item 11.)

   P(cooks L on Thursday and burns it) =

   P(cooks L on Thursday) · P(burns L|cooks L) = .909 × .1 = .0909.

   Hence P(cooks S on Friday) = .0909

   P(cooks L on Friday) = 1 - .0909 = .9091 .

2. Saturday: P(cooks L on Friday and burns it) = .9091 × .1 = .09091 .

   Hence P(cooks S on Saturday) = .09091

   P(cooks L on Saturday) = 1 - .09091 = .90909 .

   Sunday: P(cooks L on Sunday) = 1 - .90909(.1) = .909091 .

   You might have saved arithmetic by looking at the pattern:

1	Monday
.9	Tuesday
.91	Wednesday
.909	Thursday
.9091	Friday
.90909	Saturday
.909091	Sunday

3. P(burned cereal on Monday) = P(burned L) + P(burned S)

   = .9090909(.1) + .0909091(.4) = .12727273 .

4. $\frac{7}{55} = .1272727\ldots$, or $.1\overline{27}$.

Compare Exercise 3 with Item 8, Section 12-1.

Answers to Exercises 12-3

1. P(cooks L on Thursday) = .825 (Item 11).

   P(cooks S on Thursday) = 1 - .825 = .175 .

   Hence

   P(cooks L on Thursday and does not burn it) = .825 × .9 = .7425 .

   P(cooks S on Thursday and burns it) = .175 × .4 = .07 .

   P(cooks L on Friday) = .7425 + .07 = .8125

2. We have:

   P(cooks L) = .8 .

   P(cooks S) = .2 .

   P(cooks L and burns it) = .8 × .1 = .08 .

   P(cooks S and burns it) = .2 × .4 = .08 .

   The cereal is burned if she cooks L and burns it or if she cooks S and burns it. (She cannot do both.)

   Hence

   P(burned cereal) = .16 .

   This is approximately $\frac{1}{6}$.

3. $$P(L \mid \text{burned}) = \frac{P(L \cap \text{burned})}{P(\text{burned})} = \frac{.08}{.16} = \frac{1}{2} .$$

   If you find the cereal burned, it is just as likely to be Lumpies as Soggies. Look back at Question 3, Section 12-3, for another argument to support this conclusion.

   Note that we have used:

   $$P(L \mid \text{burned}) = \frac{P(L) \cdot P(\text{burned} \mid L)}{P(L) \cdot P(\text{burned} \mid L) + P(S)\, P(\text{burned} \mid S)}$$

   In other words, we have an example of Bayes' formula.

4. You might have suspected that after a while the probability of cooking Lumpies on a particular day will be about .8 , whether the cook began with Lumpies or with Soggies. Here is the reason.

   The important point about this cook is that she has a very poor memory! When she gets up in the morning, she only remembers what happened yesterday. (This is probably why her cooking doesn't improve.) If she cooks L on a particular day, subsequent events go on as though she had begun cooking that day, subject only to the probabilities of burning, which never change. Hence it doesn't matter much, in the long run, whether she begins with L or S , because after a while she is almost certain (probability nearly 1) to have cooked both L and S , and we can pretend either was on the beginning day.

If you actually computed the probability that she cooked Lumpies on Tuesday, on Wednesday, etc., you should have found the following results.

Since the cook begins with S , we see:

$P(\text{cooks } S \text{ on Monday}) = 1$,

$P(\text{cooks } L \text{ on Monday}) = 0$.

She cooks L on Tuesday only if she burns S on Monday:

$P(\text{cooks } L \text{ on Tuesday}) = .4$,

$P(\text{cooks } S \text{ on Tuesday}) = .6$.

Hence, on Tuesday:

$P(\text{cooks } L \text{ and does not burn it}) = .4 \times .9 = .36$,

$P(\text{cooks } S \text{ and burns it}) = .6 \times .4 = .24$.

From this information:

$P(\text{cooks } L \text{ on Wednesday}) = .36 + .24 = .6$.

Thus, on Wednesday:

$P(\text{cooks } L \text{ and does not burn it}) = .6 \times .9 = .54$,

$P(\text{cooks } S \text{ and burns it}) = (1 - .6) \times .4 \doteq .16$.

From this information:

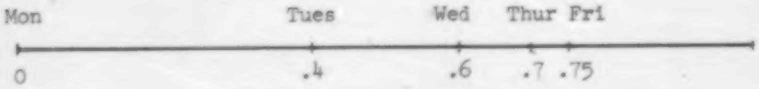
$P(\text{cooks } L \text{ on Thursday}) = .54 + .16 = .7$,

$P(\text{cooks } L \text{ on Thursday and does not burn it}) = .7 \times .9 = .63$,

$P(\text{cooks } S \text{ on Thursday and burns it}) = (1 - .7) \times .4 = .12$.

Hence:

$P(\text{cooks } L \text{ on Friday}) = .63 + .12 = .75$.



Probability of cooking L

Your guess for Saturday might be .775 . This is, in fact, the correct value of $P(\text{cooks } L \text{ on Saturday})$.

Answers to Exercises 12-4

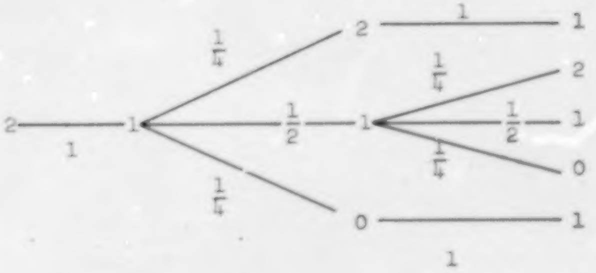
1.
$$\begin{array}{c|cc} & L & S \\ \hline L & 0 & 1 \\ S & 1 & 0 \end{array}$$

2. This cook never burns cereal. If she begins with L , she continues L . If she begins with S , she continues S .

Answers to Exercises 12-5

1. 3

2.



3.
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & 0 & 1 & 0 \end{array}$$

4. P(1 as first state) = 0 .

P(1 as second state) = 1 . (See Item 1.)

P(1 as third state) = $\frac{1}{2}$. (See Item 5.)

P(1 as fourth state) = $\frac{3}{4}$.

(P(1 as fourth state) = P(1 as third state) . $\frac{1}{2}$ + P(not 1 as third state))

5. $p_4 = \frac{3}{4}$ (Exercise 4)

$p_5 = 1 - \frac{1}{2}(\frac{3}{4}) = \frac{5}{8}$

$p_6 = 1 - \frac{1}{2}(\frac{5}{8}) = \frac{11}{16}$

$p_7 = 1 - \frac{1}{2}(\frac{11}{16}) = \frac{21}{32}$

Note that we cannot use the recursion formula to find p_7 in one step, but we can use it to find all the p's up to the one we want, step by step.

Answers to Exercises 12-6

1. $p_3 = \frac{5}{9}$ $\qquad$ $p_6 = \frac{122}{243}$

   $p_4 = \frac{14}{27}$ $\qquad$ $p_7 = \frac{365}{729}$

   $p_5 = \frac{41}{81}$ $\qquad$ $p_8 = \frac{1094}{2187}$

2. $p_{12} \approx \frac{1}{2}$. You can probably see the pattern developing in Exercise 1. However, to be sure, you would need to work out p_{12} exactly. In fact,

   $p_{12} = \frac{88574}{177147}$, and

   $$\frac{88574}{177147} - \frac{1}{2} = \frac{1}{354294} .$$